Independence of Path and Conservative Vector Fields
MATH 311, Calculus III

J. Robert Buchanan
Department of Mathematics
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Goal

We would like to know conditions on a vector field function
\( \mathbf{F}(x, y) = \langle M(x, y), N(x, y) \rangle \), so that if \( A \) and \( B \) are any two points fixed in the plane, then

\[
\int_C \mathbf{F}(x, y) \cdot d\mathbf{r}
\]

is the same regardless of the path \( C \) taken from \( A \) to \( B \).
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\[
\int_C \mathbf{F}(x, y) \cdot dr
\]

is the same regardless of the path \( C \) taken from \( A \) to \( B \).

In this case the line integral is independent of path.
A region $D \subset \mathbb{R}^2$ is called **connected** if every pair of points in $D$ can be connected by a piecewise-smooth curve lying entirely in $D$. 

**Connected**

**Not connected**
A region $D \subset \mathbb{R}^2$ is **simply connected** if every closed curve $C$ in $D$ encloses only points in $D$. 

Simply connected

Not simply connected
Theorem

Suppose that the vector field $\mathbf{F}(x, y) = \langle M(x, y), N(x, y) \rangle$ is continuous on the open, connected region $D \subset \mathbb{R}^2$. Then the line integral $\int_C \mathbf{F}(x, y) \cdot d\mathbf{r}$ is independent of path if and only if $\mathbf{F}$ is conservative on $D$. 
Suppose the vector field is conservative.

- \( \mathbf{F}(x, y) = \nabla f(x, y) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle M(x, y), N(x, y) \rangle \)

- Select any two points \( A = (x_0, y_0) \) and \( B = (x_1, y_1) \) in \( D \) and let \( C = (x(t), y(t)) \) for \( a \leq t \leq b \) be any path connecting \( A \) and \( B \).
Suppose the vector field is conservative.

- \( \mathbf{F}(x, y) = \nabla f(x, y) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \left\langle M(x, y), N(x, y) \right\rangle \)

Select any two points \( A = (x_0, y_0) \) and \( B = (x_1, y_1) \) in \( D \) and let \( C = (x(t), y(t)) \) for \( a \leq t \leq b \) be any path connecting \( A \) and \( B \).

\[
\int_C \mathbf{F}(x, y) \cdot d\mathbf{r} = \int_C M(x, y) \, dx + N(x, y) \, dy
\]

\[
= \int_a^b \left[ \frac{\partial f}{\partial x}(x(t), y(t))x'(t) + \frac{\partial f}{\partial y}(x(t), y(t))y'(t) \right] \, dt
\]

\[
= \int_a^b \frac{df}{dt}(x(t), y(t)) \, dt
\]

\[
= f(x_1, y_1) - f(x_0, y_0)
\]
Theorem

Suppose that $\mathbf{F}(x, y) = \langle M(x, y), N(x, y) \rangle$ is continuous in the open, connected region $D \subset \mathbb{R}^2$ and that $C$ is any piecewise-smooth curve lying in $D$, with an initial point $(x_0, y_0)$ and terminal point $(x_1, y_1)$. Then, if $\mathbf{F}$ is conservative on $D$, with $\mathbf{F} = \nabla f(x, y)$, we have

$$\int_C \mathbf{F}(x, y) \cdot d\mathbf{r} = f(x, y) \bigg|_{(x_0, y_0)}^{(x_1, y_1)} = f(x_1, y_1) - f(x_0, y_0).$$
Suppose $\mathbf{F}(x, y) = (2xe^{2y} + 4y^3)\mathbf{i} + (2x^2e^{2y} + 12xy^2)\mathbf{j}$.

1. Show that the line integral $\int_C \mathbf{F}(x, y) \cdot d\mathbf{r}$ is independent of path.

2. Evaluate the line integral along any piecewise-smooth curve connecting $(0, 1)$ to $(2, 3)$. 
If \( f(x, y) = x^2 e^{2y} + 4xy^3 \), then \( \nabla f(x, y) = \mathbf{F}(x, y) \) which implies \( \mathbf{F}(x, y) \) is conservative and hence independent of path.
If \( f(x, y) = x^2 e^{2y} + 4xy^3 \), then \( \nabla f(x, y) = F(x, y) \) which implies \( F(x, y) \) is conservative and hence independent of path.

By the Fundamental Theorem of Calculus for Line Integrals

\[
\int_C F(x, y) \cdot dr = f(x, y) \bigg|_{(0,1)}^{(2,3)} = 216 + 4e^6.
\]
Suppose $\mathbf{F}(x, y) = x^2 y^3 \mathbf{i} + x^3 y^2 \mathbf{j}$ and then find the work done moving from $(0, 0)$ to $(2, 1)$. 
Example (4 of 4)

\[
F(x, y) = x^2y^3 \mathbf{i} + x^3y^2 \mathbf{j}
\]

If \( f(x, y) = \frac{1}{3}x^3y^3 \), then \( \nabla f(x, y) = F(x, y) \) which implies \( F(x, y) \) is conservative and hence independent of path.
Example (4 of 4)

\[ \mathbf{F}(x, y) = x^2 y^3 \mathbf{i} + x^3 y^2 \mathbf{j} \]

1. If \( f(x, y) = \frac{1}{3} x^3 y^3 \), then \( \nabla f(x, y) = \mathbf{F}(x, y) \) which implies \( \mathbf{F}(x, y) \) is conservative and hence independent of path.

2. By the Fundamental Theorem of Calculus for Line Integrals

\[
W = \int_C \mathbf{F}(x, y) \cdot d\mathbf{r} = f(x, y) \bigg|_{(0,0)}^{(2,1)} = \frac{8}{3}.
\]
Theorem

Suppose that $\mathbf{F}(x, y)$ is continuous in the open, connected region $D \subset \mathbb{R}^2$. Then $\mathbf{F}$ is conservative on $D$ if and only if $\int_C \mathbf{F}(x, y) \cdot dr = 0$ for every piecewise-smooth closed curve lying in $D$. 
Suppose $\int_C F(x, y) \cdot dr = 0$ for every piecewise-smooth closed curve lying in $D$.

- Let $P$ and $Q$ be any two points in $D$.
- Let $C_1$ and $C_2$ be any two piecewise-smooth curves connecting $P$ and $Q$ as shown next.
\[ C = C_1 \cup (-C_2) \] is a closed curve in \( D \).
0 = \int_{C} \mathbf{F}(x, y) \cdot d\mathbf{r} \\
= \int_{C_1} \mathbf{F}(x, y) \cdot d\mathbf{r} + \int_{-C_2} \mathbf{F}(x, y) \cdot d\mathbf{r} \\
= \int_{C_1} \mathbf{F}(x, y) \cdot d\mathbf{r} - \int_{C_2} \mathbf{F}(x, y) \cdot d\mathbf{r} \\
\int_{C_1} \mathbf{F}(x, y) \cdot d\mathbf{r} = \int_{C_2} \mathbf{F}(x, y) \cdot d\mathbf{r}
Recall: a region is **simply-connected** if it contains no holes.

**Theorem**

Suppose that $M(x, y)$ and $N(x, y)$ have continuous first partial derivatives on a simply-connected region $D$. Then

\[
\int_C M(x, y) \, dx + N(x, y) \, dy \text{ is independent of path in } D \text{ if and only if } M_y(x, y) = N_x(x, y) \text{ for all } (x, y) \in D.
\]
Example

Show that the following line integral is independent of path.

\[ \int_C e^{2y} \, dx + (1 + 2xe^{2y}) \, dy \]
Example

Show that the following line integral is independent of path.

\[ \int_C e^{2y} \, dx + (1 + 2xe^{2y}) \, dy \]

Let \( M(x, y) = e^{2y} \) and \( N(x, y) = 1 + 2xe^{2y} \). Then

\[ M_y(x, y) = 2e^{2y} = N_x(x, y) \]

and thus by the previous theorem the line integral is independent of path.
Summary: suppose that $\mathbf{F}(x, y) = \langle M(x, y), N(x, y) \rangle$ where $M(x, y)$ and $N(x, y)$ have continuous first partial derivatives on an open, simply-connected region $D \subset \mathbb{R}^2$. In this case the following statements are equivalent.

1. $\mathbf{F}(x, y)$ is conservative in $D$.
2. $\mathbf{F}(x, y)$ is a gradient field in $D$ (i.e. $\mathbf{F}(x, y) = \nabla f(x, y)$ for some potential function $f$, for all $(x, y) \in D$).
3. $\int_C \mathbf{F}(x, y) \cdot d\mathbf{r}$ is independent of path in $D$.
4. $\int_C \mathbf{F}(x, y) \cdot d\mathbf{r} = 0$ for every piecewise-smooth closed curve $C$ lying in $D$.
5. $M_y(x, y) = N_x(x, y)$ for all $(x, y) \in D$. 
Suppose $F(x, y) = \frac{1}{\sqrt{x^2 + y^2}} \langle -y, x \rangle$.

The vector field is not conservative since along the closed path the vector field is always oriented with the curve (or always against it).
Suppose \( \mathbf{F}(x, y) = \langle y, x \rangle \).

The vector field is conservative since along the closed path as much of the vector field is oriented with the curve as against it.
Remark: much of what we have stated for two-dimensional vector fields can be extended to three-dimensional vector fields.

Example

Evaluate the line integral

$$\int_C y^2 \, dx + (2xy + e^{3z}) \, dy + 3ye^{3z} \, dz$$

along any piecewise-smooth path connecting \((0, 1, 1/2)\) to \((1, 0, 2)\).
If \( f(x, y, z) = xy^2 + ye^{3z} \) then \( \mathbf{F}(x, y, z) = \nabla f(x, y, z) \) which implies \( \mathbf{F}(x, y, z) \) is conservative.
\[ F(x, y, z) = \langle y^2, 2xy + e^{3z}, 3ye^{3z} \rangle \]

1. If \( f(x, y, z) = xy^2 + ye^{3z} \) then \( F(x, y, z) = \nabla f(x, y, z) \) which implies \( F(x, y, z) \) is conservative.

2. By the Fundamental Theorem of Calculus for Line Integrals

\[
\int_C F(x, y, z) \cdot dr = xy^2 + ye^{3z}\bigg|_{(0,1,1/2)}^{(1,0,2)} = -e^{3/2}.
\]
Theorem

Suppose that the vector field \( \mathbf{F}(x, y, z) \) is continuous on the open, connected region \( D \subset \mathbb{R}^3 \). Then, the line integral

\[
\int_C \mathbf{F}(x, y, z) \cdot d\mathbf{r}
\]

is independent of path in \( D \) if and only if the vector field \( \mathbf{F} \) is conservative in \( D \), that is,

\[
\mathbf{F}(x, y, z) = \nabla f(x, y, z), \quad \text{for all } (x, y, z) \text{ in } D, \quad \text{for some scalar function } f \text{ (a potential function for } \mathbf{F} \text{).}
\]

Further, for any piecewise-smooth curve \( C \) lying in \( D \), with initial point \((x_1, y_1, z_1)\) and terminal point \((x_2, y_2, z_2)\) we have

\[
\int_C \mathbf{F}(x, y, z) \cdot d\mathbf{r} = f(x, y, z) \bigg|_{(x_1, y_1, z_1)}^{(x_2, y_2, z_2)} = f(x_2, y_2, z_2) - f(x_1, y_1, z_1).
\]
Homework

- Read Section 14.3.
- Exercises: 1–51 odd