1 Introduction

(1)

(2)

(3)

(a)
(b)
(c)
(d)

(4)

(a) Linear, second order.
(b) Nonlinear, first order.
(c) Linear, second order.
(d) Nonlinear, second order.
(e) Nonlinear, second order.
(f) Nonlinear, fourth order.

(5) $a = -b^2$

(6)

(a) $u(x,t) = x\,f(t) + g(t)$ where f and g are arbitrary functions of t.
(b) $u(x,t) = x(1-t^2) + t^2$.

(7)

(a)
(b) The Laplacian operator is linear, thus the Laplacian of $v(x,y)$ is the Laplacian of $u_c(x,y)$ (which is 1) plus the Laplacian of $u(x,y)$ (which is 0).
(c) $v(x,y) = e^{2x}\sin 2y + u_3(x,y)$

(8)

(a) elliptic
(b) parabolic
(c) hyperbolic
(d) hyperbolic

(9)

(10)

(a)

$$\begin{aligned} u_t &= 0.86u_{xx} \text{ for } 0 < x < L \text{ and } t > 0 \\ u(0,t) &= u(L,t) = 10 \text{ for } t > 0 \\ u(x,0) &= 100 \text{ for } 0 < x < L \end{aligned}$$

(b)

$$\begin{aligned} u_t &= 0.86u_{xx} \text{ for } 0 < x < L \text{ and } t > 0 \\ u(0,t) &= u_x(L,t) = 0 \text{ for } t > 0 \\ u(x,0) &= 100 \text{ for } 0 < x < L \end{aligned}$$

(c)

$$\begin{aligned} u_t &= 0.86u_{xx} \text{ for } 0 < x < L \text{ and } t > 0 \\ u(0,t) &= 0 \text{ and } u(L,t) = 100 \text{ for } t > 0 \\ u(x,0) &= 100 \text{ for } 0 < x < L \end{aligned}$$

(d)
- In the first case $\lim_{t\to\infty} u(x,t) = 10$ for all $0 \le x \le L$.
- In the second case $\lim_{t\to\infty} u(x,t) = 0$ for all $0 \le x \le L$.
- In the third case $\lim_{t\to\infty} u(x,t) = 100x/L$.

(11)

(a) $U(x) = -\frac{1}{2}x^2 + x + 1 - 2L + \frac{L^2}{2}$

(b) $U(x) = \frac{h(T_L - T_0)x}{K_0 + hL} + T_0$

(c) $U(x) = T_0$

(12) $U(x) = \frac{gx^2}{2c^2} - \frac{gLx}{2c^2}$

(13)

(a)

$$\begin{aligned} u_{tt} &= c^2u_{xx} \text{ for } 0 < x < 30 \text{ and } t > 0 \\ u(0,t) &= u(30,t) = 0 \text{ for } t > 0 \\ u(x,0) &= f(x) \text{ for } 0 \le x \le 30 \\ u_t(x,0) &= 0 \text{ for } 0 \le x \le 30 \end{aligned}$$

(b)

$$\begin{aligned} u_{tt} &= c^2u_{xx} \text{ for } 0 < x < 30 \text{ and } t > 0 \\ u(0,t) &= u(30,t) = 0 \text{ for } t > 0 \\ u(x,0) &= 0 \text{ for } 0 \le x \le 30 \\ u_t(x,0) &= g(x) \text{ for } 0 \le x \le 30 \end{aligned}$$

(14)

(a)

$$\begin{aligned} X''(x) + 3X'(x) - cX(x) &= 0 \\ 4Y''(y) + cY(y) &= 0 \end{aligned}$$

(b)

$$\begin{aligned} X''(x) - cX'(x) + 3cX(x) &= 0 \\ T'(t) + cT(t) &= 0. \end{aligned}$$

(c)

$$\begin{aligned} X''(x) - cx\,X(x) &= 0 \\ Y''(y) + cy\,Y(y) &= 0. \end{aligned}$$

(d) It is not possible to separate variables in this case.

(15)

(a) This is a mathematical model of a one-dimensional rod of unit length whose ends are kept at constant temperature of 0. The initial temperature distribution along the length of the rod is given by $\sin(\pi x)$.

(b) Since the ends of the rod are kept at temperature 0 and are not insulated, in the long term all heat energy will flow out of the rod into the surrounding environment. Thus $\lim_{t\to\infty} u(x,t) = 0$.

(c) $u(x,t) = e^{-\pi^2 kt}\sin(\pi x)$

(16)

(a) $u(x,t) = e^{-\pi^2 t/16}\sin\left(\frac{\pi x}{2}\right) + 4e^{-\pi^2 t}\sin(2\pi x)$

(b) $u(x,t) = e^{-\pi^2 t/4}\sin(\pi x) - e^{-9\pi^2 t/16}\sin\left(\frac{3\pi x}{2}\right)$

(17) $u(x,t) = e^{-\pi^2 t}\sin\left(\frac{\pi x}{2}\right) - \frac{1}{2}e^{-4\pi^2 t}\sin(\pi x) + 3e^{-36\pi^2 t}\sin(3\pi x)$

(18) The eigenfunctions are members of the set $\{\cos(n\pi x/L)\}_{n=0}^{\infty}$ with corresponding eigenvalues $\{n^2\pi^2/L^2\}_{n=0}^{\infty}$.

(19) $c = n^2$, $n \in \mathbb{N}$.

(20) The eigenfunctions are members of the set $\left\{\sin\frac{(2n-1)\pi x}{2L}\right\}_{n=0}^{\infty}$ with corresponding eigenvalues $\left\{\frac{(2n-1)^2\pi^2}{(2L)^2}\right\}_{n=0}^{\infty}$.

(21) $\alpha = -a/2$, $\beta = b/2$, and $\gamma = -\alpha^2/4 + \beta^2/4 - c$.

(22) $\alpha = -b/(2a)$ and $\beta = c - b^2/(4a)$.

(23)

2 First-Order Partial Differential Equations

(1)

(2) Both Alice and Bob are correct.

(3) (a)

(b) $v(x,y) = \psi(y - c\,x)$ and $w(x, c\,x + k) = \int_0^x \phi(s, c\,s + k)\,ds$

(4)

(a) General solution: $u(x,y) = \frac{1}{2}y^2 + f(k) = \frac{1}{2}y^2 + f(xe^{-y})$ and particular solution: $u(x,y) = \frac{1}{2}y^2 + x^2e^{-2y}$

(b) General solution: $u(x,y) = y\ln x - \frac{1}{2}(\ln x)^2 + f(y - \ln x)$ and particular solution $u(x,y) = \frac{1}{2}y^2 + x^2 e^{-2y}$

(5)

(a) $u(x,y) = f(2y-3x)e^{-4x}$

(b) $u(x,y) = x\,f(xy)$

(c) $u(x,y) = f(ye^{1/x})e^{-ye^{1/x}\int \frac{1}{x}e^{-1/x}\,dx}$

(6)

(7)

(a) General solution: $u(x,y) = f(2y-5x)e^{-3x}$ and particular solution: $u(x,y) = \frac{5x-2y}{5}e^{3(5x-2y)/5}\cos\frac{5x-2y}{5}$

(b) General solution: $u(x,y) = f(2y-5x)e^{-3x}$ and particular solution: $u(x,y) = e^{\frac{2y-5x}{2}}\cos\frac{2y-5x}{2}$

(c) General solution: $u(x,y) = xf(y-2/x) + x\ln x$ and particular solution: $u(x,y) = \left(1 - \frac{xy}{2}\right)e^{\frac{2x}{2-xy}} + x\ln x - x\ln\left(\frac{2x}{2-xy}\right)$

(d) General solution: $u(x,y) = f(y/x) - 4\ln x$ and particular solution: (none)

(8) $e^{-u} = e^x - x^2 + f(B,C) = e^x - x^2 + f(y - e^x, x + e^{-z})$

(9)

(a)

(b)

(c)

(10) $u(x,y) = \psi(y - c\,x) + \int_0^x \phi(\tau, c\,\tau + s)\,d\tau$

(11)

(a) $u(x,y) = \dfrac{12}{3 - 4(2x - y - 1)}$

(b) $u(x,y) = \sin^{-1}\left(\frac{1}{2}\ln\left[\frac{x^3}{y}\right]\right)$

(c)

$$x = t + s^2, \quad y = \frac{s}{1-st}, \quad \ln|\sec u + \tan u| = t + \ln(\sec 1 + \tan 1).$$

(d) $u(x,y) = \dfrac{2}{y} + x - y$

(12)

(a)

(b)

(13) $u = F\left(\dfrac{2x}{2 + xt^2 u}\right)$

(14)

(15) $\rho_{\max} = 133.333$ cars/kilometer

(16) $q = 1000$ vehicles/hour

(17)

(18)

(19) $x = \dfrac{u_{\max} t}{3}$

(20) $t_0 = \dfrac{4}{u_{\max}\sqrt{3}}$ and $x = \dfrac{1}{\sqrt{3}}$

3 Fourier Series

(1)

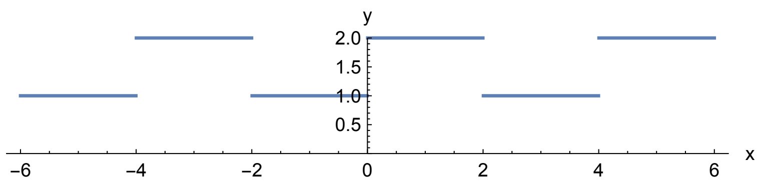
(a)

(b)

(2)

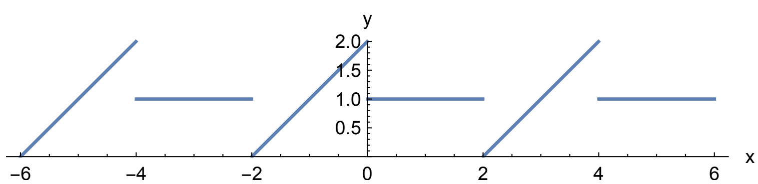
(3)

(a)



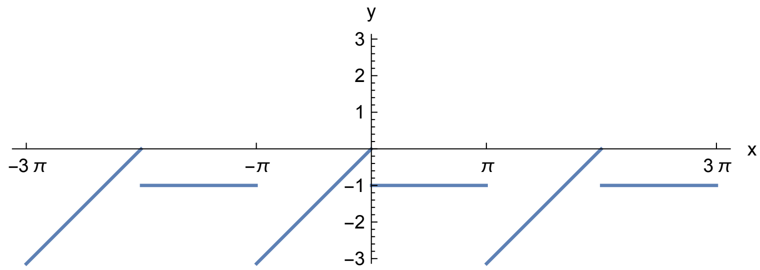
$$f(x) \sim \frac{3}{2} + \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n\pi} \sin\frac{n\pi x}{2}$$

(b)



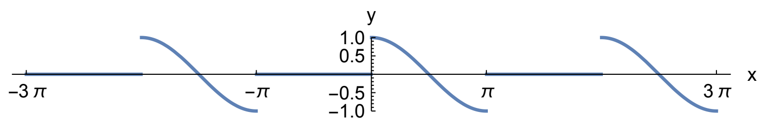
$$f(x) \sim 1 + \sum_{n=1}^{\infty} \left[\frac{2}{n^2\pi^2}(1-(-1)^n)\cos\frac{n\pi x}{2} - \frac{1}{n\pi}(1+(-1)^n)\sin\frac{n\pi x}{2}\right]$$

(c)



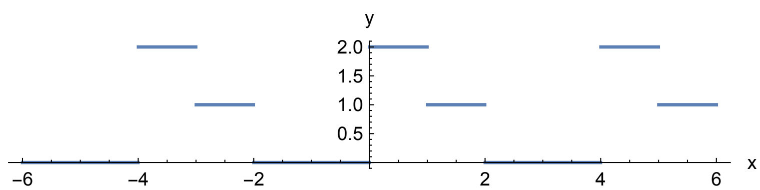
$$f(x) \sim -\frac{1}{2} - \frac{\pi}{4} + \sum_{n=1}^{\infty} \left[\frac{1}{n^2\pi}(1-(-1)^n)\cos(nx) - \frac{1+(\pi-1)(-1)^n}{n\pi}\sin(nx) \right]$$

(d)



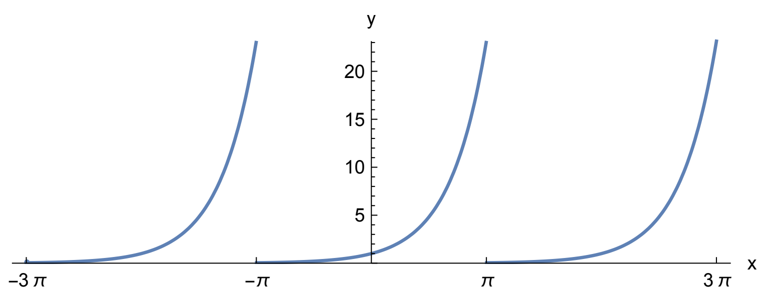
$$f(x) \sim \frac{1}{2}\cos x + \sum_{n=2}^{\infty} \frac{n(1+(-1)^n)}{\pi(n^2-1)}\sin(nx)$$

(e)



$$f(x) \sim \frac{3}{4} + \sum_{n=1}^{\infty} \left[\frac{\sin(n\pi/2)}{n\pi}\cos\frac{n\pi x}{2} + \frac{2-(-1)^n-\cos(n\pi/2)}{n\pi}\sin\frac{n\pi x}{2} \right]$$

(f)



$$f(x) \sim \frac{\sinh(a\pi)}{a\pi} + \sum_{n=1}^{\infty} \left[\frac{2(-1)^n a\sinh(a\pi)}{(a^2+n^2)\pi}\cos(nx) - \frac{2n(-1)^n\sinh(a\pi)}{(a^2+n^2)\pi}\sin(nx) \right]$$

(4)

(a)

$$a_n = \begin{cases} 1 & \text{if } n = 2 \\ 0 & \text{otherwise} \end{cases}$$

$$b_n = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

(b)

$$\begin{aligned} a_n &= 0 \text{ for all } n \\ b_n &= \begin{cases} -2 & \text{if } n = 3 \\ 2 & \text{if } n = 5 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

(c)

$$\begin{aligned} a_n &= \begin{cases} 1 & \text{if } n = 0 \\ 1/2 & \text{if } n = 2 \\ 0 & \text{otherwise} \end{cases} \\ b_n &= 0 \text{ for all } n \end{aligned}$$

(d)

$$\begin{aligned} a_n &= 0 \text{ for all } n \\ b_n &= \begin{cases} -1/2 & \text{if } n = 1 \\ 1/2 & \text{if } n = 3 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

(e)

$$f(x) = \frac{1}{2\pi} + \sin x - \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{\cos(2nx)}{4n^2 - 1}$$

(5)

$$f(x) \sim \frac{\sin(a\pi)}{a\pi} + \sum_{n=1}^{\infty} \left[\frac{2a(-1)^n \sin(a\pi)}{(a^2 - n^2)\pi} \cos(nx). \right]$$

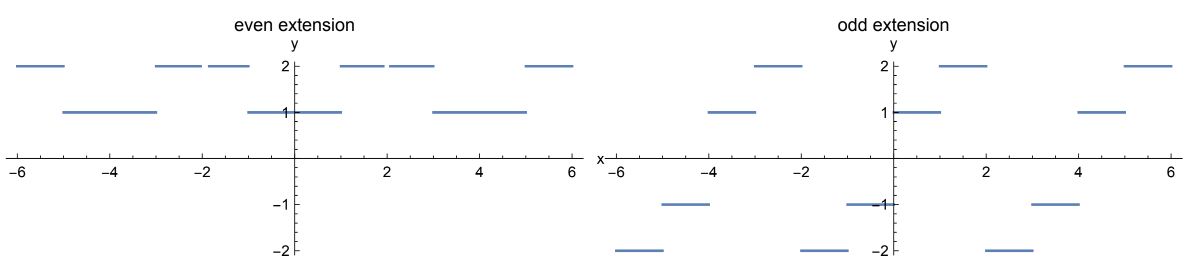
(6)

(a)

(b)

(7)

(a)



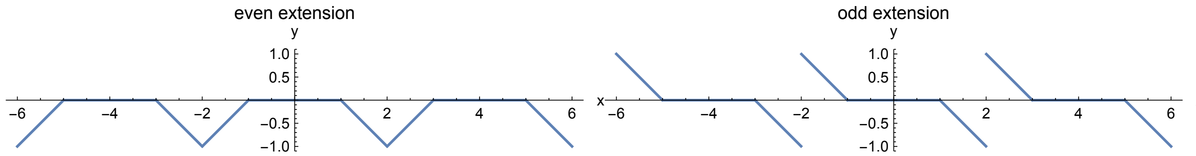
Fourier cosine series:

$$f(x) \sim \frac{3}{2} - \sum_{n=1}^{\infty} \frac{2\sin(n\pi/2)}{n\pi} \cos\frac{n\pi x}{2}$$

Fourier sine series:

$$f(x) \sim \sum_{n=1}^{\infty} \frac{4}{n\pi} \left(\cos^2\frac{n\pi}{4} - (-1)^n \right) \sin\frac{n\pi x}{2}$$

(b)



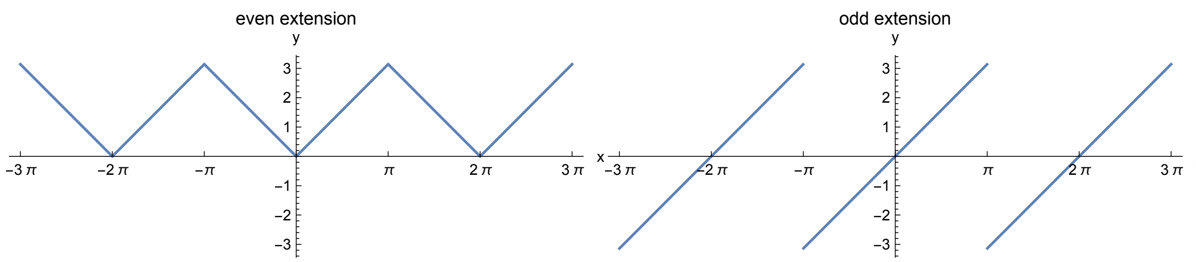
Fourier cosine series:

$$f(x) \sim -\frac{1}{4} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - (-1)^n\right)\cos\frac{n\pi x}{2}$$

Fourier sine series:

$$f(x) \sim \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}\left(n\pi(-1)^n + 2\sin\frac{n\pi}{2}\right)\sin\frac{n\pi x}{2}$$

(c)



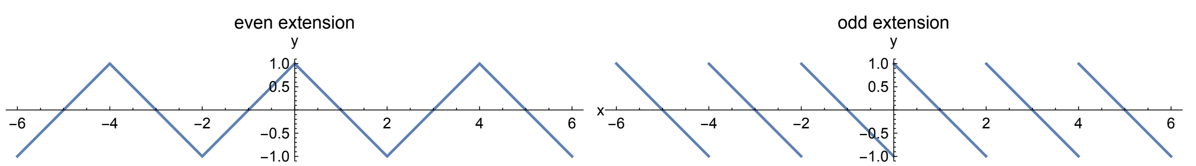
Fourier cosine series:

$$f(x) \sim \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi}(-1+(-1)^n)\cos(nx)$$

Fourier sine series:

$$f(x) \sim -\sum_{n=1}^{\infty} \frac{2(-1)^n}{n}\sin(nx)$$

(d)



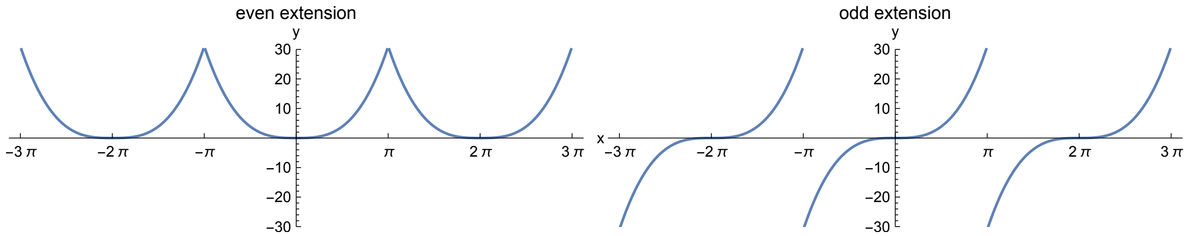
Fourier cosine series:

$$f(x) \sim \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}(1-(-1)^n)\cos\frac{n\pi x}{2}$$

Fourier sine series:

$$f(x) \sim \sum_{n=1}^{\infty} \frac{2}{n\pi}(1+(-1)^n)\sin\frac{n\pi x}{2}$$

(e)



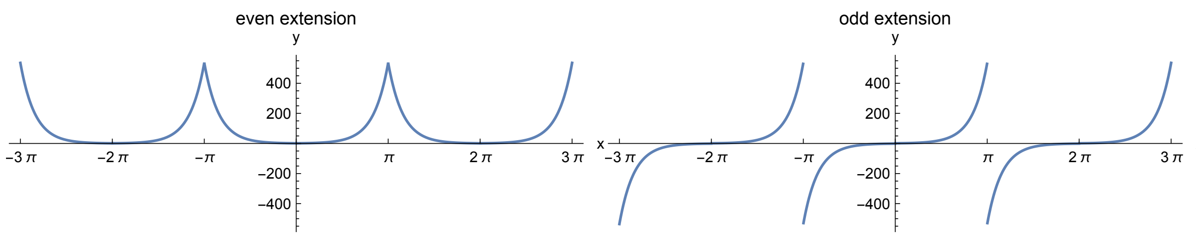
Fourier cosine series:

$$f(x) \sim \frac{\pi^3}{4} + \sum_{n=1}^{\infty} \frac{2}{n^4\pi}(6 + 3(-1)^n(n^2\pi^2 - 2))\cos(n\,x)$$

Fourier sine series:

$$f(x) \sim \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^3}(6 - n^2\pi^2)\sin(n\,x)$$

(f)



Fourier cosine series:

$$f(x) \sim \frac{e^\pi}{\pi}\sinh\pi - \sum_{n=1}^{\infty} \frac{4}{(n^2+4)\pi}(1 - (-1)^n e^{2\pi})\cos(nx)$$

Fourier sine series:

$$f(x) \sim \sum_{n=1}^{\infty} \frac{2n}{(n^2+4)\pi}(1 - (-1)^n e^{2\pi})\sin(nx)$$

(8)

$$f(x) \sim \frac{\pi}{4}(4 - \pi^2) + \sum_{n=1}^{\infty} \frac{2}{n^4\pi}(-2(n^2+3) + (-1)^n(6 + n^2(2 - 3\pi^2)))\cos(nx)$$

(9) Fourier cosine series:

$$f(x) \sim \frac{1}{a\pi}(e^{a\pi} - 1) - \sum_{n=1}^{\infty} \frac{2a}{(a^2+n^2)\pi}(1 - (-1)^n e^{a\pi})\cos(nx)$$

Fourier sine series:

$$f(x) \sim \sum_{n=1}^{\infty} \frac{2n}{(a^2+n^2)\pi}(1 - (-1)^n e^{a\pi})\sin(nx)$$

(10)

$$\hat{f}(x) = \begin{cases} f(x) & \text{if } 0 < x < L \\ -f(2L-x) & \text{if } L < x < 2L \end{cases}$$

and extend as a $4L$-periodic even function to $(-\infty, \infty)$.

$$\begin{aligned} a_0 &= 0 \\ a_{2k} &= 0 \\ a_{2k-1} &= \frac{2}{L}\int_0^L f(x)\cos\frac{(2k-1)\pi x}{2L}\,dx \end{aligned}$$

Therefore $c_0 = a_0/2 = 0$ and $c_k = a_{2k-1}$.

(11)

(a) Piecewise smooth
(b) Neither
(c) Neither
(d) Piecewise continuous
(e) Piecewise smooth
(f) Piecewise continuous
(g) Piecewise smooth
(h) Neither (undefined on $[-1, 0)$)

(12)

(a)

$$\begin{aligned} f(x) &\sim \frac{3}{2} + \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n\pi}\sin\frac{n\pi x}{2} \\ &= \begin{cases} 3/2 & \text{if } x = 4k \\ 2 & \text{if } 4k < x < 4k+2 \\ 3/2 & \text{if } x = 4k+2 \\ 1 & \text{if } 4k+2 < x < 4(k+1) \end{cases} \end{aligned}$$

where $k \in \mathbb{Z}$.

(b)

$$\begin{aligned} f(x) &\sim 1 + \sum_{n=1}^{\infty}\left[\frac{2}{n^2\pi^2}(1-(-1)^n)\cos\frac{n\pi x}{2} - \frac{1}{n\pi}(1+(-1)^n)\sin\frac{n\pi x}{2}\right] \\ &= \begin{cases} 3/2 & \text{if } x = 4k \\ 1 & \text{if } 4k < x < 4k+2 \\ 1/2 & \text{if } x = 4k+2 \\ x-(4k+2) & \text{if } 4k+2 < x < 4(k+1) \end{cases} \end{aligned}$$

where $k \in \mathbb{Z}$.

(c)

$$\begin{aligned} f(x) &\sim -\frac{1}{2} - \frac{\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{1}{n^2\pi}(1-(-1)^n)\cos(nx) - \frac{1+(\pi-1)(-1)^n}{n\pi}\sin(nx)\right] \\ &= \begin{cases} -1/2 & \text{if } x = 2k\pi \\ -1 & \text{if } 2k\pi < x < (2k+1)\pi \\ -(\pi+1)/2 & \text{if } x = (2k+1)\pi \\ x - 2(k+1)\pi & \text{if } (2k+1)\pi < x < 2(k+1)\pi \end{cases} \end{aligned}$$

where $k \in \mathbb{Z}$.

(d)

$$\begin{aligned} f(x) &\sim \frac{1}{2}\cos x + \sum_{n=2}^{\infty}\frac{n(1+(-1)^n)}{\pi(n^2-1)}\sin(nx) \\ &= \begin{cases} 1/2 & \text{if } x = 2k\pi \\ \cos x & \text{if } 2k\pi < x < (2k+1)\pi \\ -1/2 & \text{if } x = (2k+1)\pi \\ 0 & \text{if } (2k+1)\pi < x < 2(k+1)\pi \end{cases} \end{aligned}$$

where $k \in \mathbb{Z}$.

(e)

$$\begin{aligned} f(x) &\sim \frac{3}{4} + \sum_{n=1}^{\infty}\left[\frac{\sin(n\pi/2)}{n\pi}\cos\frac{n\pi x}{2} + \frac{2-(-1)^n-\cos(n\pi/2)}{n\pi}\sin\frac{n\pi x}{2}\right] \\ &= \begin{cases} 1 & \text{if } x = 4k \\ 2 & \text{if } 4k < x < 4k+1 \\ 3/2 & \text{if } x = 4k+1 \\ 1 & \text{if } 4k+1 < x < 4k+2 \\ 1/2 & \text{if } x = 4k+2 \\ 0 & \text{if } 4k+2 < x < 4(k+1) \end{cases} \end{aligned}$$

where $k \in \mathbb{Z}$.

(f)

$$\begin{aligned} f(x) &\sim \frac{\sinh(a\pi)}{a\pi} + \sum_{n=1}^{\infty}\left[\frac{2(-1)^n a\sinh(a\pi)}{(a^2+n^2)\pi}\cos(nx) - \frac{2n(-1)^n\sinh(a\pi)}{(a^2+n^2)\pi}\sin(nx)\right] \\ &= \begin{cases} e^{a(x-2k\pi)} & \text{if } (2k-1)\pi < x < (2k+1)\pi \\ \cosh(a\pi) & \text{if } x = (2k+1)\pi \end{cases} \end{aligned}$$

where $k \in \mathbb{Z}$.

(13) Fourier sine series:

$$f(x) \sim \sum_{n=1}^{\infty} \frac{2}{n^2\pi}\left(-2n(-1)^n - 2n\cos n + 3n\cos(2n) + \sin n\right)\sin(nx)$$

$$= \begin{cases} 0 & \text{if } x = -\pi \\ -2 & \text{if } -\pi < x < -2 \\ -1/2 & \text{if } x = -2 \\ 1 & \text{if } -2 < x < -1 \\ 0 & \text{if } x = -1 \\ x & \text{if } -1 < x < 1 \\ 0 & \text{if } x = 1 \\ -1 & \text{if } 1 < x < 2 \\ 1/2 & \text{if } x = 2 \\ 2 & \text{if } 2 < x < \pi \\ 0 & \text{if } x = \pi \end{cases}$$

Fourier cosine series:

$$f(x) \sim 2 - \frac{9}{2\pi} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi}\left(-1 + \cos n + 2n\sin n - 3n\sin(2n)\right)\cos(nx)$$

$$= \begin{cases} 2 & \text{if } -\pi \le x < -2 \\ -1/2 & \text{if } x = -2 \\ -1 & \text{if } -2 < x < -1 \\ 0 & \text{if } x = -1 \\ |x| & \text{if } -1 < x < 1 \\ 0 & \text{if } x = 1 \\ -1 & \text{if } 1 < x < 2 \\ 1/2 & \text{if } x = 2 \\ 2 & \text{if } 2 < x \le \pi \end{cases}$$

(14)

(15)

(16)

(17)

$$f(x) \sim \frac{3}{2} + \sum_{n=1}^{\infty} \frac{2}{(2n-1)\pi}\sin((2n-1)\pi x)$$

(18)

$$\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{1}{2} \text{ and } \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1} = \frac{1}{2} - \frac{\pi}{4}$$

(19)

(20)

(a)

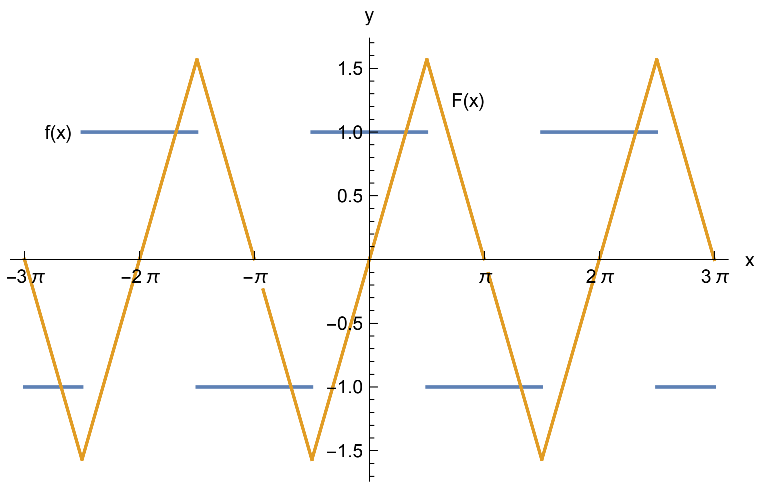
(b)

(21)

(22)

(a)

(b)



(c)

$$f(x) \sim \sum_{n=1}^{\infty} \frac{4}{n\pi} \sin \frac{n\pi}{2} \cos(nx)$$

(d)

$$F(x) \sim \sum_{n=1}^{\infty} \frac{4}{n^2\pi} \sin \frac{n\pi}{2} \sin(nx)$$

(e)

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{4}{n\pi} \sin \frac{n\pi}{2} \cos(nx) &= \begin{cases} 0 & \text{if } x = (2k-1)\pi/2 \\ 1 & \text{if } (4k-1)\pi/2 < x < (4k+1)\pi/2 \\ -1 & \text{if } (4k+1)\pi/2 < x < (4k+3)\pi/2 \end{cases} \\ \sum_{n=1}^{\infty} \frac{4}{n^2\pi} \sin \frac{n\pi}{2} \sin(nx) &= F(x) \end{aligned}$$

where $k \in \mathbb{Z}$.

(23)

(24)

(a)

$$x^2 = \sum_{k=1}^{\infty} \frac{4(-1)^{k+1}}{k^2}(1 - \cos(k\,x))$$

(b)

$$\begin{cases} x & \text{if } -\pi < x < 0 \\ 2x & \text{if } 0 < x < \pi \end{cases} = \frac{3x}{2} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^2}(1 - \cos(nx))$$

(25)

(a)

(b)

(26)

(27)

(28)

(29)

4 The Heat Equation

(1) $u(x,t) = e^{-400\pi^2 t}\sin(2\pi x) - e^{-2500\pi^2 t}\sin(5\pi x)$

(2) $u(x,t) = 1 + e^{-144\pi^2 t}\cos(3\pi x) + e^{-1024\pi^2 t}\cos(8\pi x)$

(3) $u(x,t) \sim \dfrac{200}{\pi}\displaystyle\sum_{n=1}^{\infty}\left(\cos\frac{n\pi}{4} - \cos\frac{3n\pi}{4}\right)\frac{e^{-n^2\pi^2 t}}{n}\sin(n\pi x)$

(4) $u(x,t) \sim 50 + \dfrac{200}{\pi}\displaystyle\sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{2}\right)\frac{e^{-4n^2\pi^2 t}}{n}\cos(n\pi x)$

(5) $u(x,t) = \dfrac{200}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{e^{-n^2\pi^2 t/4}}{n}\sin^2\left(\frac{n\pi}{4}\right)\sin(n\pi x)$

(6)

(a)

$$\begin{aligned} u_t &= k\,u_{xx} \text{ for } 0 < x < L \text{ and } t > 0 \\ u(0,t) &= 0 \text{ for } t > 0 \\ u_x(L,t) &= 0 \text{ for } t > 0 \\ u(x,0) &= f(x) \text{ for } 0 < x < L \end{aligned}$$

(b) $u_n(x,t) = e^{-(2n-1)^2\pi^2 kt/(4L^2)}\sin\left(\dfrac{(2n-1)\pi x}{2L}\right)$ for $n = 1, 2, \ldots$.

(c) $u(x,t) = \displaystyle\sum_{n=1}^{\infty} a_n e^{-(2n-1)^2\pi^2 kt/(4L^2)}\sin\left(\frac{(2n-1)\pi x}{2L}\right)$ where the a_n are constants chosen so that

$$f(x) \sim \sum_{n=1}^{\infty} a_n \sin\left(\frac{(2n-1)\pi x}{2L}\right).$$

(d)

$$\begin{aligned} a_n &= \frac{2}{2L}\int_0^{2L}\hat{f}(x)\sin\left(\frac{(2n-1)\pi x}{2L}\right)dx \\ &= \frac{1}{2}\int_0^1 x\sin\left(\frac{(2n-1)\pi x}{2L}\right)dx + \frac{1}{2}\int_1^3 \sin\left(\frac{(2n-1)\pi x}{2L}\right)dx + \frac{1}{2}\int_3^4 (4-x)\sin\left(\frac{(2n-1)\pi x}{2L}\right)dx. \end{aligned}$$

(7) $u(x,t) \sim \dfrac{32}{\pi^3} \displaystyle\sum_{n=1}^{\infty} \dfrac{e^{-(2n-1)^2\pi^2 t/4}}{(2n-1)^3} \sin \dfrac{(2n-1)\pi x}{2}$

(8) $u(x,t) \sim \dfrac{25}{2} + \dfrac{50}{\pi^2} \displaystyle\sum_{m=1}^{\infty} \dfrac{(-1)^m - 1}{m^2} e^{-64m^2\pi^2 t} \cos(4m\pi x)$

(9)

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} b_n e^{-(c+kn^2\pi^2/L^2)t} \sin \frac{n\pi x}{L} \\ b_n &= \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx \end{aligned}$$

(10)

(a)

(b)

$$\begin{aligned} u(x,t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} e^{-kn^2\pi^2 t/L^2} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \\ a_0 &= \frac{1}{L} \int_{-L}^{L} f(x)\, dx \\ a_n &= \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx \\ b_n &= \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx. \end{aligned}$$

(11) $u(x,t) = 20 + 30x + \dfrac{20}{\pi} \displaystyle\sum_{n=1}^{\infty} \dfrac{\left(-2 - 3(-1)^n + 8\cos\frac{n\pi}{2}\right)}{n} e^{-n^2\pi^2 t/2} \sin(n\pi x)$

(12)

(a) $U(x) = A + \dfrac{(B-A)x}{L}$

(b)

$$\begin{aligned} v_t &= kv_{xx} \quad \text{for } 0 < x < L,\ t > 0 \\ v(0,t) &= 0 \\ v(L,t) &= 0 \\ v(x,0) &= \sin \frac{\pi x}{L} - A - \frac{(B-A)x}{L} \end{aligned}$$

(c) $v(x,t) \sim \sin \dfrac{\pi x}{L} + \dfrac{2}{\pi} \displaystyle\sum_{n=1}^{\infty} \dfrac{((-1)^n B - A}{n} e^{-kn^2\pi^2 t/L^2} \sin \dfrac{n\pi x}{L}$

(d) $u(x,t) \sim A + \dfrac{(B-A)x}{L} + \sin \dfrac{\pi x}{L} + \dfrac{2}{\pi} \displaystyle\sum_{n=1}^{\infty} \dfrac{((-1)^n B - A}{n} e^{-kn^2\pi^2 t/L^2} \sin \dfrac{n\pi x}{L}$

(13)

(a) $r(x,t) = Bx^2/(2L)$.

(b)

$$\begin{aligned} v_t &= kv_{xx} + \frac{kB}{L} \text{ for } 0 < x < L \text{ and } t > 0 \\ v_x(0,t) &= 0 \\ v_x(L,t) &= 0 \\ v(x,0) &= f(x) - \frac{Bx^2}{2L} \end{aligned}$$

(c) $v(x,t) \sim \frac{kBt}{L} + \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-kn^2\pi^2 t/L^2} \cos\left(\frac{n\pi x}{L}\right)$

(d) $u(x,t) \sim \frac{Bx^2}{2L} + \frac{kBt}{L} + \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-kn^2\pi^2 t/L^2} \cos\left(\frac{n\pi x}{L}\right)$

(14)

(a) $U(x) = -\frac{x^2}{2} + \left(\frac{B-A}{L} + \frac{L}{2}\right) x + A$

(b)

$$\begin{aligned} v_t &= kv_{xx} \text{ for } 0 < x < L \text{ and } t > 0 \\ v(0,t) &= 0 \\ v(L,t) &= 0 \\ v(x,0) &= f(x) - U(x) \end{aligned}$$

(c)

$$\begin{aligned} v(x,t) &\sim \sum_{n=1}^{\infty} b_n e^{-kn^2\pi^2 t/L^2} \sin\left(\frac{n\pi x}{L}\right) \\ b_n &= \frac{2}{L}\int_0^L (f(x) - U(x)) \sin\left(\frac{n\pi x}{L}\right) dx \end{aligned}$$

(d) $u(x,t) \sim U(x) + \sum_{n=1}^{\infty} b_n e^{-kn^2\pi^2 t/L^2} \sin\left(\frac{n\pi x}{L}\right)$

(15)

(16)

(a)

(b)

(17) $u(x,t) = \frac{e^{-\frac{x^2}{1+4t}}}{\sqrt{1+4t}}$

(18) $u(x,t) = \frac{xe^{-\frac{x^2}{1+4t}}}{(1+4t)^{3/2}}$

(19) $u(x,t) = \dfrac{e^{-\frac{x^2}{1+4t}}}{\sqrt{1+4t}}$

(20)

(21)

(a)

(b)

(c) There is no contradiction since the partial differential equation is not homogeneous.

(22)

(23)

(24) $u(x,y,t) = e^{-10k\pi^2 t}\sin(\pi x)\sin(3\pi y)$

(25)

$$\begin{aligned}
U_1(x,y) &= \sum_{n=1}^{\infty} \frac{20(1-(-1)^n)}{n\pi\sinh(n\pi)}\sinh(n\pi x)\sin(n\pi y) \\
U_2(x,y) &= \sum_{n=1}^{\infty} \frac{40(1-(-1)^n)}{n\pi\sinh(n\pi)}\sin(n\pi x)\sinh(n\pi y) \\
U_3(x,y) &= \sum_{n=1}^{\infty} \frac{60(1-(-1)^n)}{n\pi\sinh(n\pi)}\sinh(n\pi(1-x))\sin(n\pi y) \\
U_4(x,y) &= \sum_{n=1}^{\infty} \frac{80(1-(-1)^n)}{n\pi\sinh(n\pi)}\sin(n\pi x)\sinh(n\pi(1-y)).
\end{aligned}$$

(26)

(a) $\lambda_1 \approx 2.02876 < \lambda_2 \approx 4.91318 < \lambda_3 \approx 7.97867 < \lambda_4 \approx 11.0855$

(b)

$$\begin{aligned}
u_1(x,t) &= e^{-2\lambda_1^2 t}\sin(\lambda_1 x) \\
u_2(x,t) &= e^{-2\lambda_2^2 t}\sin(\lambda_2 x) \\
u_3(x,t) &= e^{-2\lambda_3^2 t}\sin(\lambda_3 x) \\
u_4(x,t) &= e^{-2\lambda_4^2 t}\sin(\lambda_4 x)
\end{aligned}$$

(27) $m = -6.29218 \leq u(x,y) \leq 6.29218 = M$

5 The Wave Equation

(1) $u(x,t) = \dfrac{1}{2}\cos(2t)\sin(2x)$

(2) $u(x,t) = \cos t \sin t + 3\cos(2t)\sin(2x)$

(3) $u(x,t) = \dfrac{1}{3}\sin(3t)\sin(3x)$

(4) $u(x,t) = \cos(\pi t)\sin(\pi x) + \frac{1}{2\pi}\sin(2\pi t)\sin(2\pi x) + \frac{1}{2}\cos(3\pi t)\sin(3\pi x) + 3\cos(7\pi t)\sin(7\pi x)$

(5) $u(x,t) = -\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^4}\sin(nt)\sin(nx)$

(6) $u(x,t) = \cos(2t)\sin(2x) - \frac{32}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(1-4n^2)^2}\sin(nt)\sin(nx)$

(7) $u(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{(1-(-1)^n)}{n^2}\sin(nt) - \frac{2(1-(-1)^n)}{n^3}\cos(nt)\right]\sin(nx)$

(8) $u(x,t) = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{2}}{n^2}\cos(nt)\sin(n\pi x)$

(9)

(a)

(b)

(c) The formal solution takes the form

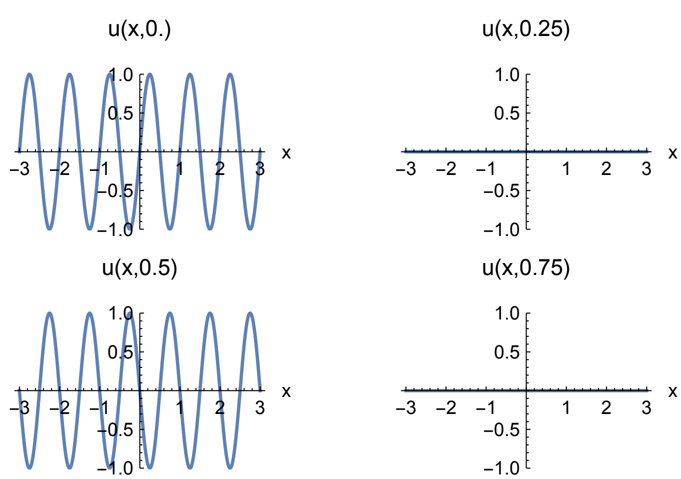
$$\begin{aligned} u(x,t) &= \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\left[2\sin\left(\frac{(2n-1)\pi}{4}\right) - \sin\left(\frac{3(2n-1)\pi}{8}\right) - \sin\left(\frac{(2n+1)\pi}{8}\right)\right] \\ &\quad * \cos\left(\frac{(2n-1)\pi ct}{2}\right)\sin\left(\frac{(2n-1)\pi x}{2}\right). \end{aligned}$$

(10)

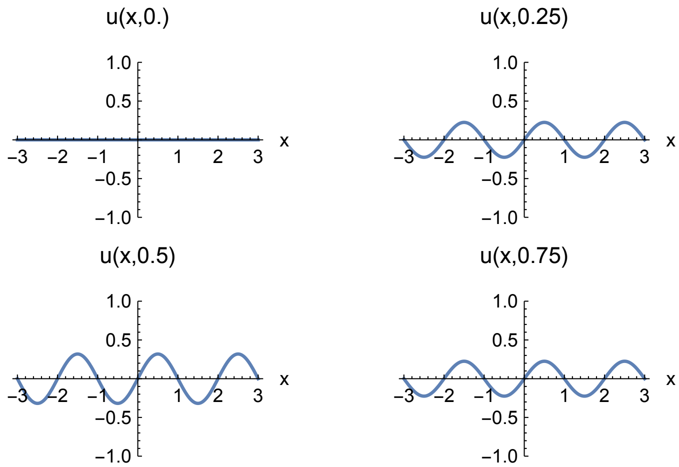
(11) $u(x,t) = \frac{1}{2}\left[e^{-(x+ct)^2} + e^{-(x-ct)^2}\right]$

(12) $u(x,t) = \frac{1}{2c}\left[\cosh(x+ct) - \cosh(x-ct)\right]$

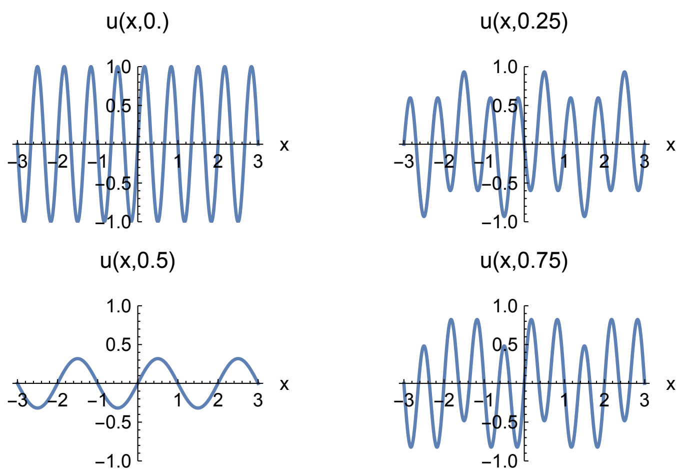
(13) $u(x,t) = \frac{1}{2}\left[\sin(2\pi(x+t)) + \sin(2\pi(x-t))\right]$



(14) $u(x,t) = \frac{1}{2\pi}\left[\cos(\pi(x-t)) - \cos(\pi(x+t))\right]$



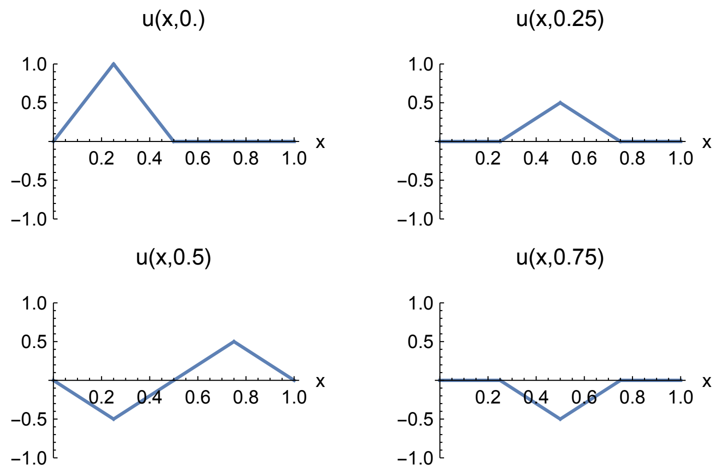
(15) $u(x,t) = \frac{1}{2}\left[\sin(3\pi(x+t)) + \sin(3\pi(x-t))\right] + \frac{1}{2\pi}\left[\cos(\pi(x-t)) - \cos(\pi(x+t))\right]$



(16) $c_{2k-1} = 0$ for $k \in \mathbb{N}$.

(17)

(a)



(b) The portion of the vibrating medium in the interval $[3/4, 1]$ is still in the initial resting position at $t = 1/4$.

(c) If $1/2 < x < 1$ the earliest time the string at position x will be displaced is $x - 1/2$.

(18) $T_m(t) = \dfrac{L}{m\pi c}\displaystyle\int_0^t f_m(s)\sin\frac{m\pi c(t-s)}{L}\,ds$

(19) $u(x,t) = \dfrac{2}{\pi^4}\displaystyle\sum_{n=1}^{\infty}\frac{(1-(-1)^n)}{n^4}(n\pi t - \sin(n\pi t))\sin(n\pi x)$

(20)

(a)

(b) $u(x,t) = \dfrac{e^{-a\,t}}{2}\left[f(x+c\,t) + f(x-c\,t)\right] + \dfrac{e^{-a\,t}}{2c}\displaystyle\int_{x-c\,t}^{x+c\,t}(a\,f(s) + g(s))\,ds$

(21) $u(x,t) = t + \dfrac{2}{\pi^2}\displaystyle\sum_{n=1}^{\infty}\frac{((-1)^n - 1)}{n^2}\sin(n\pi t)\sin(n\pi x).$

(22) $u(x,t) = \dfrac{-2g}{\pi^3}\displaystyle\sum_{n=1}^{\infty}\frac{(1-(-1)^n)}{n^3}(1-\cos(n\pi t))\sin(n\pi x)$

(23) $u(x,t) = (1-x)\sin(\pi t) - \left(t\cos(\pi t) + \dfrac{\sin(\pi t)}{\pi}\right)\sin(\pi x) + \dfrac{2}{\pi}\displaystyle\sum_{n=2}^{\infty}\frac{\sin(\pi t) - n\sin(n\pi t)}{n(n^2-1)}\sin(n\pi x)$

(24) $E'(t) = 0$

(25)

(26)

6 The Laplace Equation

(1)

(a) Many answers are possible.

(b) Many answers are possible.

(c) Many answers are possible.

(2) $u(x,y) = \sum_{n=1}^{\infty} \frac{2(1-(-1)^n)}{n\pi \sinh(n\pi)} \sinh(n\pi x)\sin(n\pi y)$

(3) $u(x,y) = \sum_{n=1}^{\infty} \frac{4(1-(-1)^n)}{n^3\pi^3 \sinh(n\pi)} \sinh(n\pi x)\sin(n\pi y)$

(4) $u(x,y) = \sum_{n=1}^{\infty} \frac{200(1-(-1)^n)}{n\pi \sinh(n\pi)} \sin(n\pi x)(\sinh(n\pi(1-y)) + \sin(n\pi y))$

(5) $u(x,y) = \frac{\sin(\pi x)\sinh(\pi(1-y))}{\sinh \pi} + \sum_{n=1}^{\infty} \frac{2(1-(-1)^n)}{n\pi \sinh(n\pi)} \sin(n\pi x)\sinh(n\pi y))$

(6) $u(x,y) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi \sinh(n\pi)} \sin(n\pi x)\sinh(n\pi(1-y)) + \sum_{n=1}^{\infty} \frac{2}{n\pi \sinh(n\pi)} \sinh(n\pi x)\sin(n\pi y)$

(7)

$$\begin{aligned} u(x,y) &= \sum_{n=1}^{\infty} \frac{2(1-(-1)^n)}{n\pi \sinh(n\pi)} \sin(n\pi x)\sinh(n\pi(1-y)) + \sum_{n=1}^{\infty} \frac{2(1-(-1)^n)}{n\pi \sinh(n\pi)} \sin(n\pi x)\sinh(n\pi y) \\ &\quad - \frac{4}{3\pi \sinh(\pi)} \sinh(\pi x)\sin(\pi y) + \sum_{n=3}^{\infty} \frac{2n(1-(-1)^n)}{\pi(n^2-4)\sinh(n\pi)} \sinh(n\pi x)\sin(n\pi y) \end{aligned}$$

(8)

(9)

(10) $v(r,\theta) = r^2\cos(2\theta)$

(11) $v(r,\theta) = \frac{r^3}{8}\sin(3\theta)$

(12) $v(r,\theta) = \frac{2\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^{n+1} r^n}{n^2}\cos(n\theta)$

(13) $v(r,\theta) = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{r^n}{n\pi}\left[\sin\left(\frac{n\pi}{2}\right)\cos(n\theta) + \left(1-\cos\left(\frac{n\pi}{2}\right)\right)\sin(n\theta)\right]$

(14) $v(r,\theta) = \sum_{n=1}^{\infty} \left(\frac{r}{2}\right)^n \left[\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)\cos(n\theta) + \frac{1}{n\pi}\left(1-(-1)^n\right)\sin(n\theta)\right]$

(15) $v(r,\theta) = \sum_{n=1}^{\infty} \left(\frac{r}{2}\right)^n \left[\left(\frac{2((-1)^n-1)}{n^2\pi^2} + \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)\right)\cos(n\theta) + \frac{4}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\sin(n\theta)\right]$

(16) $v(r,\theta) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left(1-\cos\frac{n\pi}{2}\right) r^n \sin(n\theta)$

(17) $v(r,\theta) = \sum_{n=1}^{\infty} \frac{r^{2n/3}}{n^2}\left(3\sin\frac{2n\pi}{3} - 2n\pi\cos\frac{2n\pi}{3}\right)\sin\left(\frac{2n\theta}{3}\right)$

(18) $v(r,\theta)=\sum_{n=1}^{\infty}\frac{2}{(4^n-1)n\pi}\left(1-\cos\frac{n\pi}{2}\right)\left(-r^n+4^n r^{-n}\right)\sin(n\theta)$

(19) $v(r,\theta)=\sum_{n=1}^{\infty}(a_n^+ r^{2n/3}+a_n^- r^{-2n/3})\sin\left(\frac{2n\theta}{3}\right)$ where

$$\begin{aligned} a_n^+ &= \frac{-2(1-(-1)^n)n-2^{(2n+3)/3}n\pi^2\cos\frac{2n\pi}{3}+2^{2n/3}(3\pi)\sin\frac{2n\pi}{3}}{n^2\pi(4^{n/3}-1)} \\ a_n^- &= \frac{2^{2n/3}\left(2^{(2n+3)/3}(1-(-1)^n)n+2n\pi^2\cos\frac{2n\pi}{3}-3\pi\sin\frac{2n\pi}{3}\right)}{n^2\pi(4^{n/3}-1)}. \end{aligned}$$

(20) $u(x,y)=\frac{\cos(\pi x)\cosh(\pi y)}{\pi\sinh\pi}$

(21) $u(x,y)=\sum_{n=1}^{\infty}\frac{-2(1-(-1)^n)}{n^3\pi^3\sinh(2n\pi)}\cosh(n\pi x)\cos(n\pi y)$

(22) $v(r,\theta)=a_0+r\sin\theta$

(23) $v(r,\theta)=a_0+\frac{3}{5}\left(\frac{r}{3}\right)^5\cos(5\theta)$

(24) $u(x,y)=\sum_{n=1}^{\infty}\frac{-2((2n-1)\pi+4(-1)^n)}{n\pi^3(2n-1)^2\cosh(2n\pi)}\sinh(n\pi x)\sin\left(\frac{(2n-1)\pi y}{2}\right)$

(25) $u(x,y)=\sum_{n=1}^{\infty}\frac{8(2(-1)^n+(2n-1)\pi)}{\pi^2(2n-1)^2\cosh(3n\pi)}\cosh(n\pi x)\sin\left(\frac{(2n-1)\pi(2-y)}{4}\right)$

(26) $u(x,y)=x^2+y^2-a^2$

(27) $u(x,y)=\frac{1}{12}x^3y+\frac{1}{12}xy^3-\frac{1}{2}a^2xy$

(28)

$$\begin{aligned} u(x,y) &= x^2+\sum_{n=1}^{\infty}\frac{2}{\pi\sinh n\pi}\left(\frac{(-1)^n\pi^2}{n}-\frac{2((-1)^n-1)}{n^3}\right)\sinh(n(\pi-y))\sin(nx) \\ &\quad +\sum_{n=1}^{\infty}\frac{2((-1)^n-1)\pi}{n\sinh n\pi}\sinh(nx)\sin(ny) \\ &\quad +\sum_{n=1}^{\infty}\frac{2}{\pi\sinh n\pi}\left(\frac{(-1)^n\pi^2}{n}-\frac{2((-1)^n-1)}{n^3}\right)\sin(nx)\sinh(ny). \end{aligned}$$

(29)

(30)

(31)

(32)

(33)

(34)

7 Sturm-Liouville Theory

(1) $y(x) = -x^2/2 + 3x/2 + 2$

(2) $y(x) = -\frac{1}{6}x^3 + \frac{(6+\pi^3+3\pi^2)x}{6(\pi+1)} + 1$

(3) No nontrivial solutions.

(4) No nontrivial solutions.

(5) $[x\, y']' + \frac{1}{x}\, y = 0$

(6) $\left[e^{bx/a}\, y'\right]' + \frac{1}{a}e^{bx/a} y = 0$

(7)

(8)

(a)

(b)

(9)

(a) $9x^2 - 66x + 119$

(b) 0

(c) 0

(d) $18\cosh(3x) - 18\sinh(3x)$

(10)

(11)

(12)

(13)

(14) $\lambda_n = (2n-1)^2\pi^2/(4L^2)$ and $y_n(x) = B_n \sin((2n-1)\pi x/(2L))$

(15) $\phi_n(x) = -Bn\pi\cos(n\pi x) + B\sin(n\pi x)$ is an eigenfunction corresponding to the eigenvalue $\lambda_n = n^2\pi^2$.

(16) The eigenvalues are $\lambda_n = \gamma_n^2$ where γ_n is the nth root of the Bessel function of the first kind of order 0. The corresponding eigenfunction is $\phi_n(x) = J_0(\gamma_n x)$.

(17)

(18)

(19)

(a)

(b)

(c)

(d)

(20)

(21)

(a)
(b)
(c)

(22)

(23)

(24) $f(x) = 0$.

(25)

(26)

(27)

(28) Various answers are possible.

(29) Various answers are possible.

(30)

8 Special Functions

(1)

(2)

(3) $-2\sqrt{\pi}$

(4)

(5)

(6)

(a)
(b)

(7)

(a) $\frac{1}{3}$
(b) $\frac{1}{4}$
(c) $\frac{\pi}{16}$
(d) $\frac{2}{15}$

(8)

(a)
(b)

(9) $\sqrt{2\pi}\,\dfrac{\Gamma(1/4)}{\Gamma(3/4)}$

(10)

(11)

(12)

(13)

(a) $f(x) \sim 0.769756J_0(\lambda_{0,1}x){+}0.661472J_0(\lambda_{0,2}x){-}0.282963J_0(\lambda_{0,3}x){-}0.46433J_0(\lambda_{0,4}x){+}0.198712J_0(\lambda_{0,5}x)+\cdots$

(b) $f(x) \sim 1.14652J_2(\lambda_{2,1}x){-}0.875544J_2(\lambda_{2,2}x){+}0.74048J_2(\lambda_{2,3}x){-}0.654457J_2(\lambda_{2,4}x){+}0.593202J_2(\lambda_{2,5}x)+\cdots$

(c) $f(x) \sim 1.05095J_3(\lambda_{3,1}x){-}0.821503J_3(\lambda_{3,2}x){+}0.703991J_3(\lambda_{3,3}x){-}0.627577J_3(\lambda_{3,4}x){+}0.572301J_3(\lambda_{3,5}x)+\cdots$

(d) $f(x) \sim 0.982109J_4(\lambda_{4,1}x){-}0.779533J_4(\lambda_{4,2}x){+}0.674312J_4(\lambda_{4,3}x){-}0.605009J_4(\lambda_{4,4}x){+}0.554342J_4(\lambda_{4,5}x)+\cdots$

(14) $\lambda_{3/2,1} \approx 4.49341,\ \lambda_{3/2,2} \approx 7.72525,\ \lambda_{3/2,3} \approx 10.9041,\ \lambda_{3/2,4} \approx 14.0662,\ \lambda_{3/2,5} \approx 17.2208$

(15)

(16)

$$\begin{aligned} j_0(x) &= \frac{\sin x}{x} \\ j_1(x) &= \frac{\sin x}{x^2} - \frac{\cos x}{x} \\ j_2(x) &= \frac{3\sin x}{x^3} - \frac{3\cos x}{x^2} - \frac{\sin x}{x} \end{aligned}$$

(17)

(a)

(b)

(c)

(d)

(18)

(19)

(20)

(21)

(a) 0

(b) $-\dfrac{4}{15}$

(c) $\dfrac{n!}{2^{n-1}} \displaystyle\sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)k!(n-k)!}$

(d) 0

(22)

(a) $\begin{cases} -\frac{8}{9}+\frac{1}{3}\ln 4 & \text{if } n=1 \\ \frac{2}{2-n-n^2} & \text{if } n>1 \end{cases}$

(b) $\begin{cases} -\frac{8}{9}+\frac{1}{3}\ln 4 & \text{if } n=1 \\ \frac{2(-1)^n}{n^2+n-2} & \text{if } n>1 \end{cases}$

(23)

(a) $f(x) \sim \sum_{n=1}^{\infty} \frac{(2n+1)\sqrt{\pi}}{2\Gamma(1-n/2)\Gamma((3+n)/2)} P_n(x)$

(b) $f(x) \sim \sum_{n=0}^{\infty} \frac{(2n+1)\sqrt{\pi}}{8\Gamma(2+n/2)\Gamma((3-n)/2)} P_n(x)$

(c) $f(x) \sim \sum_{n=0}^{\infty} \frac{(4n+1)\sqrt{\pi}}{4(n+1)!\Gamma(3/2-n)} P_{2n}(x)$

(d) $f(x) \sim \sum_{k=1}^{\infty} a_{2k-1}P_{2k-1}(x)$

(24)

(a) $\frac{x}{2}(1-x^2)^{1/2}$

(b) $\frac{1}{2}(5x^3-3x)$

(c) $\frac{15}{2}(x^2-1)(1-7x^2)$

(d) $\frac{1}{48}(x^2-1)(1-7x^2)$

(25) $f(x) \sim 0.126651P_2^2(x) - 0.0954058P_4^2(x) - 0.0133263P_6^2(x) - 0.00680953P_8^2(x) - 0.00353408P_{10}^2(x) + \cdots$

(26) $f(x) \sim -\frac{9\pi}{32}P_1^1(x) + \frac{7\pi}{256}P_3^1(x) + \frac{11\pi}{4096}P_5^1(x) + \frac{45\pi}{65536}P_7^1(x) + \frac{133\pi}{524288}P_9^1(x) + \cdots$

(27)

(28)

(29) $f(x) = L_0(x)$

(30) $e^{-1}\sum_{n=0}^{\infty} \frac{L_n(x)}{n!}$

(31) $\left[x^{\alpha+1}e^{-x}y'\right]' + nx^{\alpha}e^{-x}y = 0$

(32) $\frac{\Gamma(5+\alpha)}{24\Gamma(1+\alpha)} - \frac{\Gamma(5+\alpha)}{6\Gamma(2+\alpha)}x + \frac{\Gamma(5+\alpha)}{4\Gamma(3+\alpha)}x^2 - \frac{\Gamma(5+\alpha)}{6\Gamma(4+\alpha)}x^3 + \frac{x^4}{24}$

(33)

(34)

(35)

(36)

(37)

(38) $\dfrac{\Gamma(\alpha+2)}{\Gamma(\alpha+1)} - 1 - \alpha + x$

(39)

(40)

(41)

(42) $\dfrac{3}{2}H_0(x) + \dfrac{1}{2}H_1(x) + \dfrac{1}{4}H_2(x)$

(43) $f(x) \sim \dfrac{1}{e^{1/4}} \displaystyle\sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}H_{2n-1}(x)}{2^{2n-1}(2n-1)!}$

(44) $f(x) \sim e^{1/4} \displaystyle\sum_{n=0}^{\infty} \dfrac{1}{2^n n!} H_n(x)$

(45)

(46)

(47)

(a)

(b)

(c)

(48)

(a)

(b)

(49)

(50)

(51)

(a) $\dfrac{1}{2}H_0(x) + \dfrac{1}{2}H_2(x)$

(b) $\dfrac{3}{4}H_1(x) + \dfrac{1}{4}H_3(x)$

(c) $\dfrac{3}{8}H_0(x) + \dfrac{1}{2}H_2(x) + \dfrac{1}{8}H_4(x)$

(d) $\dfrac{4}{\pi} \displaystyle\sum_{n=1}^{\infty} \dfrac{T_{2n-1}(x)}{(2n-1)^2}$

9 Applications of PDEs in the Physical Sciences

(1)

(2)

(3)

(4)

(5)

(a)

(b)

(6) $u(x,t) = 16L^2\sqrt{\frac{L}{g}}\sum_{n=1}^{\infty}\frac{1}{\lambda_{0,n}^4 J_1(\lambda_{0,n})}\sin\left(\frac{\lambda_{0,n}}{2}\sqrt{\frac{g}{L}}t\right)J_0\left(\lambda_{0,n}\sqrt{1-\frac{x}{L}}\right)$

(7) $u(x,t) = e^{-\gamma t/2}\sum_{n=1}^{\infty}(A_n\cos(\mu_n t) + B_n\sin(\mu_n t))J_0\left(\lambda_{0,n}\sqrt{1-\frac{x}{L}}\right)$ where

$$\begin{aligned} A_n &= \frac{2}{(J_0(\lambda_{0,n}))^2}\int_0^1 f(L(1-\xi^2))J_0(\lambda_{0,n}\xi)\xi\,d\xi \\ B_n &= \frac{\gamma A_n}{\mu_n} + \frac{4}{\mu_n(J_0(\lambda_{0,n}))^2}\int_0^1 g(L(1-\xi^2))J_0(\lambda_{0,n}\xi)\xi\,d\xi. \end{aligned}$$

(8)

(9)

(10) $u(\rho,\varphi) = \frac{100r_1 - 30r_2}{r_1 - r_2} - \frac{70r_1r_2}{(r_1-r_2)\rho} + 25\left(\frac{r_1^2r_2^2(r_1+r_2)}{(r_1^3-r_2^3)\rho^2} - \frac{(r_1^2+r_2^2)\rho}{r_1^3-r_2^3}\right)\cos\varphi$

(11) $u(r,\theta,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left[J_m\left(\frac{\lambda_{m,n}r}{r_0}\right)(A_{m,n}\cos(m\theta) + B_{m,n}\sin(m\theta))\sin\left(\frac{c\lambda_{m,n}t}{r_0}\right)\right]$ where

$$\begin{aligned} A_{0,n} &= \frac{2r_0}{c\lambda_{0,n}\pi(J_1(\lambda_{0,n}))^2}\int_0^1 z\,J_0(\lambda_{0,n}z)\left(\int_{-\pi}^{\pi} g(r_0z,\theta)\,d\theta\right)dz \\ A_{m,n} &= \frac{2r_0}{c\lambda_{m,n}\pi(J_{m+1}(\lambda_{m,n}))^2}\int_0^1 z\,J_m(\lambda_{m,n}z)\left(\int_{-\pi}^{\pi} g(r_0z,\theta)\cos(m\theta)\,d\theta\right)dz \\ B_{m,n} &= \frac{2r_0}{c\lambda_{m,n}\pi(J_{m+1}(\lambda_{m,n}))^2}\int_0^1 z\,J_m(\lambda_{m,n}z)\left(\int_{-\pi}^{\pi} g(r_0z,\theta)\sin(m\theta)\,d\theta\right)dz \end{aligned}$$

(12) $u(r,\theta,t) = v(r,\theta,t) + w(r,\theta,t)$ where

$$v(r,\theta,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left[J_m\left(\frac{\lambda_{m,n}r}{r_0}\right)(A_{m,n}\cos(m\theta) + B_{m,n}\sin(m\theta))\cos\left(\frac{c\lambda_{m,n}t}{r_0}\right)\right]$$

with

$$\begin{aligned}
A_{0,n} &= \frac{2}{\pi(J_1(\lambda_{0,n}))^2}\int_0^1 z\,J_0(\lambda_{0,n}z)\left(\int_{-\pi}^{\pi} f(r_0 z,\theta)\,d\theta\right)dz\\
A_{m,n} &= \frac{2}{\pi(J_{m+1}(\lambda_{m,n}))^2}\int_0^1 z\,J_m(\lambda_{m,n}z)\left(\int_{-\pi}^{\pi} f(r_0 z,\theta)\cos(m\theta)\,d\theta\right)dz\\
B_{m,n} &= \frac{2}{\pi(J_{m+1}(\lambda_{m,n}))^2}\int_0^1 z\,J_m(\lambda_{m,n}z)\left(\int_{-\pi}^{\pi} f(r_0 z,\theta)\sin(m\theta)\,d\theta\right)dz
\end{aligned}$$

and

$$w(r,\theta,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left[J_m\left(\frac{\lambda_{m,n}r}{r_0}\right)(\hat{A}_{m,n}\cos(m\theta)+\hat{B}_{m,n}\sin(m\theta))\sin\left(\frac{c\lambda_{m,n}t}{r_0}\right)\right]$$

with

$$\begin{aligned}
\hat{A}_{0,n} &= \frac{2r_0}{c\lambda_{0,n}\pi(J_1(\lambda_{0,n}))^2}\int_0^1 z\,J_0(\lambda_{0,n}z)\left(\int_{-\pi}^{\pi} g(r_0 z,\theta)\,d\theta\right)dz\\
\hat{A}_{m,n} &= \frac{2r_0}{c\lambda_{m,n}\pi(J_{m+1}(\lambda_{m,n}))^2}\int_0^1 z\,J_m(\lambda_{m,n}z)\left(\int_{-\pi}^{\pi} g(r_0 z,\theta)\cos(m\theta)\,d\theta\right)dz\\
\hat{B}_{m,n} &= \frac{2r_0}{c\lambda_{m,n}\pi(J_{m+1}(\lambda_{m,n}))^2}\int_0^1 z\,J_m(\lambda_{m,n}z)\left(\int_{-\pi}^{\pi} g(r_0 z,\theta)\sin(m\theta)\,d\theta\right)dz.
\end{aligned}$$

(13)

(14)

(15)

(16)

(17) $u(\varphi,\theta,t) = -\sin(3\theta)\sum_{k=2}^{\infty}\left[(4k+1)\frac{(2k-3)!}{(2k+3)!}\cos(\sqrt{2k(2k+1)}t)P_{2k}^{3}(\cos\varphi)\right]$

(18) $u(\varphi,\theta,t) = \sum_{n=1}^{\infty}\sum_{m=-n}^{n} b_{n,m}\sin(\sqrt{n(n+1)}t)Y_n^m(\varphi,\theta)$ where

$$b_{n,m} = \sqrt{\frac{2n+1}{4\pi n(n+1)}\frac{(n-m)!}{(n+m)!}}\int_0^{\pi/2}\int_0^{\pi/2}\varphi\,\theta\left(\frac{\pi}{2}-\varphi\right)\left(\frac{\pi}{2}-\theta\right)e^{-im\theta}P_n^m(\cos\varphi)\sin\varphi\,d\theta\,d\varphi.$$

(19)

(20)

(21) $T = 0$

(22)

(23)

(24)

(25)

(a)

(b)

(c)

(d)

(26)

(a)

(b)

(27) The total number of stationary states is n^2.

(28)

(29)

(30) $p = \dfrac{(e^2-5)}{e^2} \approx 0.323324$

10 Nonhomogeneous Initial Boundary Value Problems

(1)

(2) $\dfrac{d}{dt}\displaystyle\int_0^t e^{s\,t}\cos t\,ds = 2e^{t^2}\cos t + \frac{1-e^{t^2}}{t^2}\cos t + \frac{1-e^{t^2}}{t}\sin t$

(3) $\dfrac{d}{dt}\displaystyle\int_0^t \tan^{-1}(s^2t)(1+st)^2\,ds = \tan^{-1}(t^3)(1+t^2)^2 + \int_0^t \left[\frac{s^2(1+st)^2}{1+s^4t^2} + 2s(1+st)\tan^{-1}(s^2t)\right] ds$

(4) $u(x,t) = \dfrac{1}{9}(1-e^{-9t})\sin(3x)$

(5) $u(x,t) = \left(\dfrac{t^2}{9} - \dfrac{2t}{81} + \dfrac{2}{729} - \dfrac{2}{729}e^{-9t}\right)\sin(3x)$

(6) $u(x,t) = \dfrac{1}{9}(1-e^{-9t})\sin(3x) + \dfrac{8}{\pi^2}\displaystyle\sum_{n=1}^{\infty}\frac{e^{-(2n-1)^2t}}{(2n-1)^3}\sin((2n-1)x)$

(7) $u(x,t) = \left(\dfrac{t^2}{9} - \dfrac{2t}{81} + \dfrac{2}{729} - \dfrac{2}{729}e^{-9t}\right)\sin(3x) + \dfrac{4}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{n(1+(-1)^n e^{-\pi})e^{-n^2t}}{n^4+4}\sin(nx)$

(8)

$$\begin{aligned} u(x,t) &= (e^{-t}-1)x/\pi + 1 + \frac{16}{\pi}\sum_{n=1}^{\infty}\frac{e^{-(2n-1)^2t}}{(2n-1)((2n-1)^2-4)}\sin((2n-1)x) \\ &\quad + \frac{1}{9}(1-e^{-9t})\sin(3x) + \frac{2te^{-t}}{\pi}\sin x - \frac{2}{\pi}\sum_{n=2}^{\infty}\frac{(-1)^n}{n(n^2-1)}\left(e^{-t}-e^{-n^2t}\right)\sin(nx) \end{aligned}$$

(9)

$$\begin{aligned} u(x,t) &= (1-\cos t-\sin t)x/\pi + \sin t + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{n(1+(-1)^n e^{-\pi})}{n^4+4}e^{-n^2t}\sin(nx) \\ &\quad + \frac{81t^2-18t+2-2e^{-9t}}{729}\sin(3x) + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(n^2+(-1)^n)(e^{-n^2t}-\cos t)+((-1)^n n^2-1)\sin t}{n(n^4+1)}\sin(nx) \end{aligned}$$

(10) $T_2(t) = \frac{4}{17\pi}\sin t + \frac{(20-17\pi)e^{-4t}}{51\pi} + \frac{(\pi-1)e^{-t}}{3\pi} - \frac{1}{17\pi}\cos t$

(11)

$$\begin{aligned} u(x,t) &= (1-\cos t)\frac{x^2}{2\pi} + 1 - e^{-t}\cos x + \frac{t+t^2-\sin t}{\pi} - \frac{\pi}{6}(1-\cos t) + \frac{e^{-t}+\sin t-\cos t}{\pi}\cos x \\ &\quad - \frac{2}{\pi}\sum_{n=2}^{\infty}\Big(e^{-n^2t}((-1)^n(2n^4-n^2+1)+n^4+1)+(1+(-1)^n)(n^4+1)(n^2t-1) \\ &\quad + (-1)^n n^2(n^2-1)(n^2\sin t-\cos t)\Big)\frac{\cos(nx)}{n^4(n^4+1)(n^2-1)} \end{aligned}$$

(12)

$$\begin{aligned} u(x,t) &= \sin t + \frac{16}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n(1-e^{-(2n-1)^2t/4})}{(4n^2-1)(2n-3)}\cos\left(\frac{(2n-1)x}{2}\right) \\ &\quad + \frac{16}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n\left(4\sin t+(2n-1)^2\left[\cos t - e^{-(2n-1)^2t/4}\right]\right)}{(2n-1)(8n(n-1)(2n^2-2n+1)+17)}\cos\left(\frac{(2n-1)x}{2}\right) \end{aligned}$$

(13)

$$\begin{aligned} u(x,t) &= \frac{e^{x+t}}{\sqrt{2\pi}}\int_{-\infty}^{-(x+2t)/\sqrt{2t}} e^{-z^2/2}\,dz + \frac{e^{-x+t}}{\sqrt{2\pi}}\int_{-(x-2t)/\sqrt{2t}}^{\infty} e^{-z^2/2}\,dz \\ &\quad + \frac{1}{2}\left(\cos(x+t)+\sin(x+t)-e^{-t}(\cos x+\sin x)\right) \end{aligned}$$

(14)

$$\begin{aligned} u(x,t) &= \frac{e^{-x+t}}{\sqrt{2\pi}}\int_{-(x-2t)/\sqrt{2t}}^{\infty} e^{-z^2/2}\,dz + \frac{e^{x+t}}{\sqrt{2\pi}}\int_{(x+2t)/\sqrt{2t}}^{\infty} e^{-z^2/2}\,dz \\ &\quad + \int_0^t \frac{e^{-x+t-2s}}{\sqrt{2\pi}}\int_{(-x+2(t-s))/\sqrt{2(t-s)}}^{\infty} e^{-z^2/2}\,dz\,ds + \int_0^t \frac{e^{x+t-2s}}{\sqrt{2\pi}}\int_{(x+2(t-s))/\sqrt{2(t-s)}}^{\infty} e^{-z^2/2}\,dz\,ds \end{aligned}$$

(15) $u(x,t) = \sin x\sin t + \frac{t}{2}\cos(x-t) - \frac{1}{2}\cos x\sin t$

(16)

$$\begin{aligned} u(x,t) &= \cos x\cos t + x^2t + \frac{1}{3}t^3 + \frac{1}{4}(e^{-x+t}-e^{-x})(\cos(x-t)+\sin(x-t)) \\ &\quad - \frac{1}{20}e^{-x}(3\cos(x-t)-\sin(x-t)) - \frac{1}{20}e^{-x-t}(\sin(x+t)-3\cos(x+t)) \end{aligned}$$

(17)

(18) $u(x,t) = \frac{t}{2}\sin t\sin x + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(\sin(2nt)-2n\sin t)}{(4n^2-1)^2}\sin(2nx)$

(19)

$$
\begin{aligned}
u(x,t) &= (-t+\sin t)\frac{x}{\pi} + \left(\left[\frac{1}{\pi}+\frac{t}{2}\right]\sin t - \frac{t}{\pi}\cos t\right)\sin x \\
&\quad + \sum_{n=2}^{\infty} \frac{(2(2n-1)\sin t - \sin((2n-1)t))}{(2n-1)^2((2n-1)^2-1)\pi}\sin((2n-1)x) \\
&\quad + \sum_{n=1}^{\infty}\left(\frac{2(\sin(2nt)-2n\sin t)}{4n^2(4n^2-1)\pi} + \frac{4(\sin(2nt)-2n\sin t)}{(4n^2-1)^2\pi}\right)\sin(2nx)
\end{aligned}
$$

(20)

$$
\begin{aligned}
u(x,t) &= (-t+\sin t)\frac{x}{\pi} + \cos(2t)\sin(2x) + \left(\left[\frac{1}{\pi}+\frac{t}{2}\right]\sin t - \frac{t}{\pi}\cos t\right)\sin x \\
&\quad + \sum_{n=2}^{\infty} \frac{(2(2n-1)\sin t - \sin((2n-1)t))}{(2n-1)^2((2n-1)^2-1)\pi}\sin((2n-1)x) \\
&\quad + \sum_{n=1}^{\infty}\left(\frac{2(\sin(2nt)-2n\sin t)}{4n^2(4n^2-1)\pi} + \frac{4(\sin(2nt)-2n\sin t)}{(4n^2-1)^2\pi}\right)\sin(2nx).
\end{aligned}
$$

11 Nonlinear Partial Differential Equations

(1) The bounded traveling wave solutions are constant solutions.

(2) $u(x,t) = A\cos\dfrac{x-c\,t}{\sqrt{c^2-\alpha^2}} + B\sin\dfrac{x-c\,t}{\sqrt{c^2-\alpha^2}}$ where A and B are arbitrary constants.

(3) If $c < \beta/\alpha$,

$$u(x,t) = A\cos\left((\beta-\alpha\,c)^{1/2}(x-c\,t)\right) + B\sin\left((\beta-\alpha\,c)^{1/2}(x-c\,t)\right) + \frac{A_0}{\alpha\,c-\beta}.$$

If $\alpha\,c \geq \beta$, the only bounded traveling wave solution is the constant solution, which is a special case of the solution above.

(4)

(a) $\omega^2 = \alpha^2 k^2 + 1$

(b) $\omega^2 = \alpha^2 k^4$

(c) $\omega = k(\alpha - \beta k^2)$

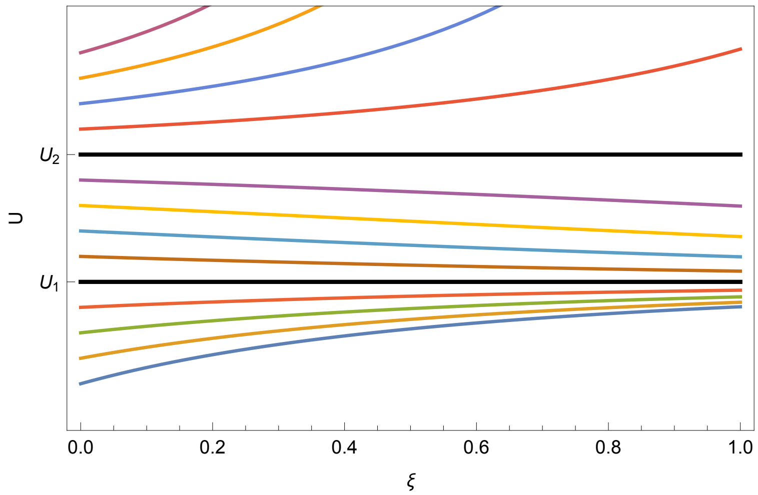
(5) $U'' + cU' = \begin{cases} U & \text{if } U < 1/2 \\ 0 & \text{if } U > 1/2 \end{cases}$

(6)

(a)

(b)

(7) Many results are possible.



(8)

(9)

$$U' = \frac{1}{\nu}(F(U) - c\,U + A)$$

where A is an arbitrary constant. Separating the variables produces the ordinary differential equation,

$$\frac{1}{F(U) - c\,U + A}\,dU = \frac{1}{\nu}\,d\xi.$$

As long as the denominator of the left-hand side does not vanish, this equation can be integrated and an implicit solution for $U(\xi)$ (and hence $u(x,t)$) can be found. If there exists U for which $F(U) - c\,U + A = 0$, this implies $F''(U) = 0$ which contradicts the assumption that $F''(U) > 0$ for all U.

(10) $U'' + c\,U' + U - U^2 = 0$

(11)

(12)

(13)

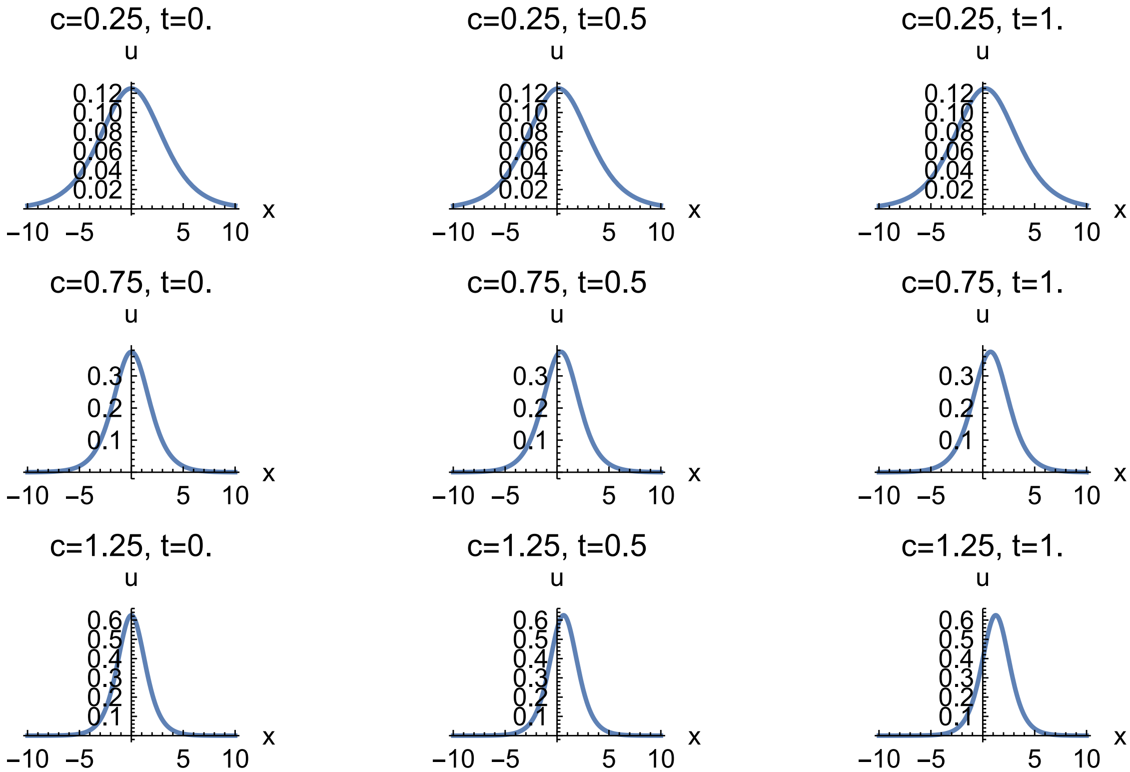
(a) $\left(\frac{dU}{d\xi}\right)^2 = \frac{2}{1-c^2}(A - \cos U)$

(b) $U(\xi) = 4\tan^{-1}\left(e^{\pm(\xi-\xi_0)/\sqrt{1-c^2}}\right)$

(14)

(15)

(16)



(17)

(18)

(19)

(20)

(21)

$$-\beta U^{(4)} + (c^2 - \alpha^2 - 2\gamma U)U'' - 2\gamma(U')^2 = 0$$

12 Numerical Solutions to PDEs Using Finite Differences

(1)

(a) $f'(1) \approx 1.38857$

(b) $E \leq 0.0119567$

(2)

(a) $f'(1) \approx 1.37667$

(b) $E \leq 0.00528763$

(3)

(a) $f''(1) \approx 0.238035$

(b) $E \leq 0.00148454$

(4)

(5)

(6)

(7)

(8)

t	$u(0,t)$	$u(0.1,t)$	$u(0.2,t)$	$u(0.3,t)$	$u(0.4,t)$	$u(0.5,t)$	$u(0.6,t)$	$u(0.7,t)$	$u(0.8,t)$	$u(0.9,t)$	$u(1,t)$
0.000	0.0000	5.8779	19.0211	28.5317	23.5114	0.0000	-35.2671	-66.5740	-76.0845	-52.9007	0.0000
0.001	0.0000	6.7946	18.8844	27.1235	21.3770	-1.7634	-35.5368	-64.8025	-72.6783	-49.1681	0.0000
0.002	0.0000	7.5130	18.7026	25.7499	19.3487	-3.3958	-35.7164	-63.0349	-69.3834	-45.8756	0.0000
0.003	0.0000	8.0677	18.4707	24.4115	17.4229	-4.9041	-35.8126	-61.2746	-66.2262	-42.9450	0.0000
0.004	0.0000	8.4859	18.1880	23.1081	15.5959	-6.2946	-35.8317	-59.5277	-63.2196	-40.3163	0.0000
0.005	0.0000	8.7894	17.8560	21.8389	13.8641	-7.5735	-35.7799	-57.8012	-60.3680	-37.9428	0.0000
0.006	0.0000	8.9957	17.4781	20.6032	12.2237	-8.7468	-35.6637	-56.1016	-57.6702	-35.7874	0.0000
0.007	0.0000	9.1191	17.0585	19.4002	10.6711	-9.8203	-35.4893	-54.4347	-55.1219	-33.8202	0.0000
0.008	0.0000	9.1717	16.6015	18.2293	9.2026	-10.7997	-35.2631	-52.8052	-52.7169	-32.0171	0.0000
0.009	0.0000	9.1636	16.1119	17.0898	7.8148	-11.6905	-34.9910	-51.2167	-50.4479	-30.3582	0.0000
0.010	0.0000	9.1032	15.5941	15.9816	6.5040	-12.4981	-34.6788	-49.6718	-48.3072	-28.8269	0.0000

(9)

t	$u(0,t)$	$u(0.1,t)$	$u(0.2,t)$	$u(0.3,t)$	$u(0.4,t)$	$u(0.5,t)$	$u(0.6,t)$	$u(0.7,t)$	$u(0.8,t)$	$u(0.9,t)$	$u(1,t)$
0.000	1.0000	1.0000	1.0000	1.0000	1.0000	2.0000	2.0000	2.0000	2.0000	2.0000	2.0000
0.001	1.0000	1.0000	1.0000	1.0000	1.1100	1.9100	2.0000	2.0000	2.0000	2.0000	2.0000
0.002	1.0000	1.0000	1.0000	1.0121	1.1881	1.8479	1.9919	2.0000	2.0000	2.0000	2.0000
0.003	1.0000	1.0000	1.0013	1.0304	1.2448	1.8044	1.9798	1.9993	2.0000	2.0000	2.0000
0.004	1.0000	1.0001	1.0044	1.0513	1.2871	1.7733	1.9662	1.9976	1.9999	2.0000	2.0000
0.005	1.0000	1.0006	1.0092	1.0731	1.3194	1.7508	1.9523	1.9950	1.9997	2.0000	2.0000
0.006	1.0001	1.0015	1.0154	1.0944	1.3446	1.7341	1.9388	1.9917	1.9993	2.0000	2.0000
0.007	1.0004	1.0029	1.0229	1.1148	1.3650	1.7216	1.9262	1.9878	1.9987	1.9999	2.0000
0.008	1.0009	1.0049	1.0312	1.1341	1.3817	1.7120	1.9146	1.9834	1.9979	1.9998	2.0000
0.009	1.0017	1.0074	1.0401	1.1520	1.3957	1.7045	1.9039	1.9788	1.9968	1.9997	1.9999
0.010	1.0028	1.0105	1.0495	1.1688	1.4078	1.6987	1.8942	1.9741	1.9955	1.9994	1.9999

(10)

t	$u(0,t)$	$u(0.1,t)$	$u(0.2,t)$	$u(0.3,t)$	$u(0.4,t)$	$u(0.5,t)$	$u(0.6,t)$	$u(0.7,t)$	$u(0.8,t)$	$u(0.9,t)$	$u(1,t)$
0.00	0.0000	0.3090	0.5878	0.8090	0.9511	1.0000	0.9511	0.8090	0.5878	0.3090	0.0000
0.01	0.0000	0.2859	0.5437	0.7484	0.8798	0.9250	0.8798	0.7484	0.5437	0.2859	0.0000
0.02	0.0000	0.2644	0.5030	0.6923	0.8138	0.8557	0.8138	0.6923	0.5030	0.2644	0.0000
0.03	0.0000	0.2446	0.4653	0.6404	0.7528	0.7915	0.7528	0.6404	0.4653	0.2446	0.0000
0.04	0.0000	0.2263	0.4304	0.5924	0.6964	0.7322	0.6964	0.5924	0.4304	0.2263	0.0000
0.05	0.0000	0.2093	0.3981	0.5480	0.6442	0.6773	0.6442	0.5480	0.3981	0.2093	0.0000
0.06	0.0000	0.1936	0.3683	0.5069	0.5959	0.6265	0.5959	0.5069	0.3683	0.1936	0.0000
0.07	0.0000	0.1791	0.3407	0.4689	0.5512	0.5796	0.5512	0.4689	0.3407	0.1791	0.0000
0.08	0.0000	0.1657	0.3151	0.4337	0.5099	0.5361	0.5099	0.4337	0.3151	0.1657	0.0000
0.09	0.0000	0.1532	0.2915	0.4012	0.4716	0.4959	0.4716	0.4012	0.2915	0.1532	0.0000
0.10	0.0000	0.1418	0.2696	0.3711	0.4363	0.4587	0.4363	0.3711	0.2696	0.1418	0.0000

(11)

t	$u(0,t)$	$u(0.1,t)$	$u(0.2,t)$	$u(0.3,t)$	$u(0.4,t)$	$u(0.5,t)$	$u(0.6,t)$	$u(0.7,t)$	$u(0.8,t)$	$u(0.9,t)$	$u(1,t)$
0.00	-1.0000	-1.0000	-1.0000	-1.0000	-1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.01	-1.0112	-1.0022	-0.9576	-0.7889	-0.1624	0.1625	0.7890	0.9579	1.0033	1.0153	1.0178
0.02	-0.9903	-0.9583	-0.8351	-0.5549	-0.2378	0.2381	0.5557	0.8374	0.9657	1.0129	1.0244
0.03	-0.9289	-0.8804	-0.7305	-0.4971	-0.1641	0.1655	0.5003	0.7391	0.9031	0.9857	1.0095
0.04	-0.8493	-0.8004	-0.6593	-0.4314	-0.1530	0.1574	0.4401	0.6793	0.8454	0.9427	0.9741
0.05	-0.7712	-0.7257	-0.5927	-0.3882	-0.1315	0.1419	0.4059	0.6280	0.7944	0.8947	0.9282
0.06	-0.6983	-0.6562	-0.5350	-0.3475	-0.1159	0.1353	0.3769	0.5870	0.7466	0.8458	0.8793
0.07	-0.6312	-0.5929	-0.4821	-0.3114	-0.0994	0.1304	0.3545	0.5509	0.7027	0.7980	0.8304
0.08	-0.5699	-0.5349	-0.4338	-0.2776	-0.0836	0.1281	0.3355	0.5190	0.6619	0.7523	0.7832
0.09	-0.5137	-0.4818	-0.3892	-0.2463	-0.0679	0.1271	0.3194	0.4903	0.6241	0.7092	0.7383
0.10	-0.4622	-0.4329	-0.3481	-0.2168	-0.0528	0.1273	0.3054	0.4643	0.5892	0.6689	0.6962

(12)

t	$u(0,t)$	$u(0.1,t)$	$u(0.2,t)$	$u(0.3,t)$	$u(0.4,t)$	$u(0.5,t)$	$u(0.6,t)$	$u(0.7,t)$	$u(0.8,t)$	$u(0.9,t)$	$u(1,t)$
0.00	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.01	0.9143	0.9817	0.9961	0.9991	0.9997	0.9995	0.9986	0.9956	0.9863	0.9572	0.8666
0.02	0.8968	0.9569	0.9862	0.9957	0.9978	0.9966	0.9916	0.9791	0.9505	0.8947	0.8245
0.03	0.8793	0.9438	0.9757	0.9895	0.9925	0.9884	0.9763	0.9519	0.9105	0.8528	0.7776
0.04	0.8680	0.9321	0.9663	0.9810	0.9829	0.9744	0.9546	0.9216	0.8744	0.8128	0.7416
0.05	0.8580	0.9219	0.9562	0.9702	0.9693	0.9558	0.9299	0.8911	0.8397	0.7778	0.7083
0.06	0.8485	0.9116	0.9448	0.9567	0.9524	0.9344	0.9036	0.8608	0.8073	0.7454	0.6784
0.07	0.8385	0.9003	0.9317	0.9410	0.9332	0.9112	0.8768	0.8313	0.7767	0.7156	0.6507
0.08	0.8274	0.8878	0.9169	0.9234	0.9124	0.8871	0.8499	0.8027	0.7477	0.6876	0.6250
0.09	0.8151	0.8738	0.9006	0.9044	0.8904	0.8625	0.8232	0.7750	0.7202	0.6614	0.6010
0.10	0.8014	0.8584	0.8830	0.8842	0.8678	0.8377	0.7970	0.7483	0.6941	0.6366	0.5783

(13)

t	$u(0,t)$	$u(0.1,t)$	$u(0.2,t)$	$u(0.3,t)$	$u(0.4,t)$	$u(0.5,t)$	$u(0.6,t)$	$u(0.7,t)$	$u(0.8,t)$	$u(0.9,t)$	$u(1,t)$
0.00	0.0000	0.8100	2.5600	4.4100	5.7600	6.2500	5.7600	4.4100	2.5600	0.8100	0.0000
0.01	1.1930	1.5760	2.7122	4.1203	5.2463	5.6994	5.3168	4.1789	2.6103	1.1658	0.5314
0.02	1.8595	2.1430	2.9266	3.9619	4.8516	5.2368	4.9322	3.9813	2.6651	1.4690	0.9541
0.03	2.3407	2.5383	3.1095	3.8760	4.5561	4.8591	4.6067	3.8145	2.7159	1.7232	1.2994
0.04	2.6847	2.8311	3.2512	3.8220	4.3332	4.5565	4.3372	3.6754	2.7600	1.9352	1.5844
0.05	2.9391	3.0471	3.3589	3.7830	4.1615	4.3155	4.1174	3.5611	2.7978	2.1118	1.8206
0.06	3.1273	3.2075	3.4385	3.7518	4.0266	4.1236	3.9401	3.4683	2.8302	2.2589	2.0168
0.07	3.2667	3.3260	3.4962	3.7248	3.9191	3.9705	3.7977	3.3942	2.8585	2.3817	2.1801
0.08	3.3693	3.4126	3.5365	3.7003	3.8323	3.8480	3.6839	3.3356	2.8836	2.4845	2.3162
0.09	3.4438	3.4749	3.5633	3.6773	3.7613	3.7496	3.5931	3.2899	2.9063	2.5710	2.4300
0.10	3.4968	3.5186	3.5798	3.6556	3.7026	3.6702	3.5208	3.2547	2.9271	2.6440	2.5254

(14)

t	$u(0,t)$	$u(0.1,t)$	$u(0.2,t)$	$u(0.3,t)$	$u(0.4,t)$	$u(0.5,t)$	$u(0.6,t)$	$u(0.7,t)$	$u(0.8,t)$	$u(0.9,t)$	$u(1,t)$
0.00	0.0000	0.3600	0.6400	0.8400	0.9600	1.0000	0.9600	0.8400	0.6400	0.3600	0.0000
0.05	0.0000	0.3905	0.6776	0.8455	0.9206	0.9400	0.9206	0.8455	0.6776	0.3905	0.0000
0.10	0.0000	0.3951	0.6853	0.8277	0.8673	0.8703	0.8673	0.8277	0.6853	0.3951	0.0000
0.15	0.0000	0.3735	0.6561	0.7843	0.8048	0.7991	0.8048	0.7843	0.6561	0.3735	0.0000
0.20	0.0000	0.3292	0.5883	0.7139	0.7358	0.7307	0.7358	0.7139	0.5883	0.3292	0.0000
0.25	0.0000	0.2674	0.4871	0.6176	0.6600	0.6649	0.6600	0.6176	0.4871	0.2674	0.0000
0.30	0.0000	0.1936	0.3636	0.4993	0.5749	0.5966	0.5749	0.4993	0.3636	0.1936	0.0000
0.35	0.0000	0.1140	0.2315	0.3659	0.4763	0.5175	0.4763	0.3659	0.2315	0.1140	0.0000
0.40	0.0000	0.0352	0.1037	0.2266	0.3604	0.4177	0.3604	0.2266	0.1037	0.0352	0.0000
0.45	0.0000	-0.0352	-0.0105	0.0900	0.2254	0.2893	0.2254	0.0900	-0.0105	-0.0352	0.0000
0.50	0.0000	-0.0907	-0.1058	-0.0379	0.0725	0.1289	0.0725	-0.0379	-0.1058	-0.0907	0.0000

(15) $-r^2u_{i-1}^{j+1} + 2(1+r^2)u_i^{j+1} - r^2u_{i+1}^{j+1} = r^2u_{i-1}^{j} + 2(2-r^2)u_i^{j} + r^2u_{i+1}^{j} - 2u_i^{j-1}$

(16)

(17)

k	$u_1^{(k)}$	$u_2^{(k)}$	$u_3^{(k)}$	$u_4^{(k)}$	$u_5^{(k)}$	$u_6^{(k)}$	$u_7^{(k)}$	$u_8^{(k)}$
0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1	-1.7500	0.1667	0.8000	0.8000	0.8333	0.2000	-0.5000	1.6000
2	-1.5917	0.7306	0.8933	1.1200	1.4000	0.5267	0.1000	1.8267
3	-1.7093	0.7764	0.8286	1.1731	1.5611	0.6587	0.2367	2.1240
4	-1.7370	0.7924	0.8252	1.2738	1.6307	0.6779	0.3243	2.1942
5	-1.7418	0.7907	0.8321	1.3129	1.6619	0.6912	0.3670	2.2458
6	-1.7397	0.7925	0.8360	1.3386	1.6803	0.6988	0.3897	2.2683
7	-1.7391	0.7922	0.8378	1.3507	1.6903	0.7033	0.4017	2.2817
8	-1.7386	0.7924	0.8390	1.3579	1.6956	0.7056	0.4081	2.2885
9	-1.7384	0.7924	0.8395	1.3615	1.6985	0.7069	0.4116	2.2923
10	-1.7382	0.7924	0.8398	1.3635	1.7000	0.7076	0.4134	2.2943
11	-1.7381	0.7924	0.8400	1.3646	1.7009	0.7080	0.4145	2.2954
12	-1.7381	0.7924	0.8401	1.3652	1.7013	0.7082	0.4150	2.2960
13	-1.7381	0.7924	0.8401	1.3655	1.7016	0.7083	0.4153	2.2963
14	-1.7381	0.7924	0.8402	1.3657	1.7017	0.7083	0.4155	2.2965

(18)

k	$u_1^{(k)}$	$u_2^{(k)}$	$u_3^{(k)}$	$u_4^{(k)}$	$u_5^{(k)}$	$u_6^{(k)}$	$u_7^{(k)}$	$u_8^{(k)}$
0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1	-1.7500	0.4583	0.7083	0.6167	1.0542	0.5525	-0.3458	1.8650
2	-1.6875	0.7569	0.7489	1.0119	1.5297	0.6557	0.2192	2.1522
3	-1.7520	0.7946	0.8159	1.2624	1.6477	0.6927	0.3536	2.2527
4	-1.7447	0.7936	0.8323	1.3334	1.6852	0.7035	0.3965	2.2830
5	-1.7403	0.7928	0.8378	1.3558	1.6967	0.7069	0.4097	2.2924
6	-1.7388	0.7925	0.8395	1.3628	1.7003	0.7079	0.4138	2.2954
7	-1.7383	0.7924	0.8400	1.3649	1.7014	0.7083	0.4151	2.2963
8	-1.7381	0.7924	0.8401	1.3656	1.7017	0.7084	0.4155	2.2965
9	-1.7381	0.7924	0.8402	1.3658	1.7018	0.7084	0.4156	2.2966

(19) The linear system can be expressed as $A\,\mathbf{u} = \mathbf{b}$ where matrix A is as shown in Eq. (12.29). The vector $\mathbf{b} = \mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3$ where

$$\mathbf{b}_1 = \left(f_1^1, \ldots, f_{N-1}^1, f_1^2, \ldots, f_{N-1}^2, \ldots, f_1^{M-1}, \ldots, f_{N-1}^{M-1}\right)^T$$

$$\mathbf{b}_2 = \frac{h^2}{12}\left((\Delta f)_1^1, \ldots, (\Delta f)_{N-1}^1, (\Delta f)_1^2, \ldots, (\Delta f)_{N-1}^2, \ldots, (\Delta f)_1^{M-1}, \ldots, (\Delta f)_{N-1}^{M-1}\right)^T$$

and $\mathbf{b}_3 = \mathbf{g}_4 + \mathbf{g}$ where $\mathbf{g}_4$ and $\mathbf{g}$ may be found in Eqs. (12.30) and (12.31) respectively.

(20) Choose $N = M = 10$ which implies $h = 1/10$. The homogeneous Neumann boundary condition along the edge where $x = 1$ implies that $u^j_{N+1} = u^j_{N-1}$ for $j = 1, 2, \ldots, M-1$. Likewise homogeneous Neumann boundary condition along the edge where $y = 1$ implies that $u_i^{M+1} = u_i^{M-1}$ for $i = 1, 2, \ldots, N-1$. With $\epsilon = 10^{-6}$ the Gauss-Seidel method requires 400 iterations from an initial approximation of $\mathbf{u}^{(0)} = \mathbf{0}$ to converge to the solution in the table below.

	0	1	2	3	4	5	6	7	8	9	10
0	2.7183	2.4596	2.2255	2.0138	1.8221	1.6487	1.4918	1.3499	1.2214	1.1052	1.0000
1	2.4596	2.2851	2.1105	1.9445	1.7899	1.6482	1.5198	1.4056	1.3077	1.2312	1.1907
2	2.2255	2.1105	1.9871	1.8636	1.7450	1.6342	1.5335	1.4452	1.3726	1.3213	1.3003
3	2.0138	1.9445	1.8636	1.7780	1.6923	1.6102	1.5348	1.4690	1.4163	1.3810	1.3680
4	1.8221	1.7899	1.7450	1.6923	1.6359	1.5796	1.5265	1.4798	1.4426	1.4183	1.4097
5	1.6487	1.6482	1.6342	1.6102	1.5796	1.5457	1.5117	1.4809	1.4561	1.4399	1.4342
6	1.4918	1.5198	1.5335	1.5348	1.5265	1.5117	1.4939	1.4761	1.4611	1.4510	1.4474
7	1.3499	1.4056	1.4452	1.4690	1.4798	1.4809	1.4761	1.4685	1.4610	1.4555	1.4536
8	1.2214	1.3077	1.3726	1.4163	1.4426	1.4561	1.4611	1.461	1.4588	1.4566	1.4557
9	1.1052	1.2312	1.3213	1.3810	1.4183	1.4399	1.4510	1.4555	1.4566	1.4564	1.4562
10	1.0000	1.1907	1.3003	1.3680	1.4097	1.4342	1.4474	1.4536	1.4557	1.4562	1.4562

(21)

(22)

(23)

(24) $\dfrac{1}{2\sin^2\frac{\pi}{2N}} \le r^2 \le \dfrac{3}{2\sin^2\frac{(N-1)\pi}{2N}}$