American Options

An Undergraduate Introduction to Financial Mathematics

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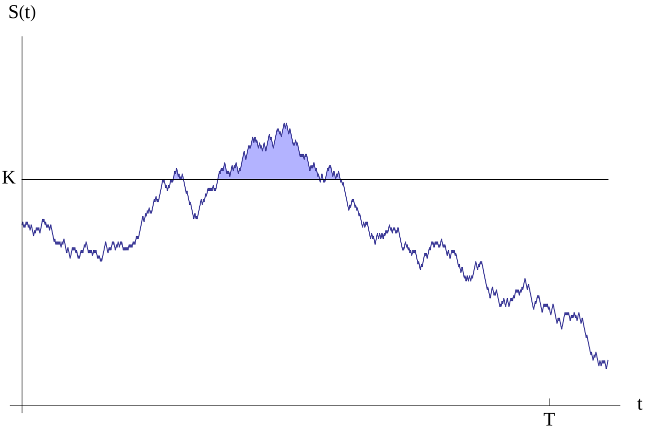
Early Exercise

Since American style options give the holder the same rights as European style options *plus* the possibility of early exercise we know that

$$C^e \leq C^a \quad \text{and} \quad P^e \leq P^a.$$

Early Exercise

An American option (a call, for instance) may have a positive payoff even when the corresponding European call has zero payoff.



Trade-offs of Early Exercise

Consider an American-style call option on a dividend-paying stock. If the option is exercised early you,

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- pay the strike price K early foregoing the interest

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you could have earned on it, and
- lose the insurance provided by the call in case $S(T) < K$.

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Theorem

Suppose the current value of a security is S, the risk-free interest rate is r, and C^a and P^a are the values of an American call and put respectively on the security with strike price K and expiry $T > 0$. Then

$$C^a + K \geq S + P^a$$

Proof

Assume to the contrary that $C^a + K < S + P^a$.

- Sell the security, sell the put, and buy the call. This produces a cash flow of $S + P^a - C^a$.
- Invest this amount at the risk-free rate,
- If the owner of the American put chooses to exercise it at time $0 \leq t \leq T$, the call option can be exercised to purchase the security for K.
- The net balance of the investment is

$$(S + P^a - C^a)e^{rt} - K > Ke^{rt} - K \geq 0.$$

- If the American put expires out of the money, exercise the call to close the short position in the security at time T. The net balance of the investment is

$$(S + P^a - C^a)e^{rT} - K > Ke^{rT} - K > 0.$$

Thus the investor receives a non-negative profit in either case, violating the principle of no arbitrage.

Another Inequality

Theorem

Suppose the current value of a security is S, the risk-free interest rate is r, and C^a and P^a are the values of an American call and put respectively on the security with strike price K and expiry $T > 0$. Then

$$S + P^a \geq C^a + Ke^{-rT}$$

Suppose $S + P^a < C^a + Ke^{-rT}$.

- Sell an American call and buy the security and the American put. Thus $C^a - S - P^a$ is borrowed at $t = 0$.
- If the owner of the call decides to exercise it at any time $0 \leq t \leq T$, sell the security for the strike price K by exercising the put. The amount of loan to be repaid is $(C^a - S - P^a)e^{rt}$ and

$$\begin{aligned} (C^a - S - P^a)e^{rt} + K &= (C^a + Ke^{-rt} - S - P^a)e^{rt} \\ &\geq (C^a + Ke^{-rT} - S - P^a)e^{rt} \end{aligned}$$

since $r > 0$. By assumption $S + P_a < C^a + Ke^{-rT}$, so the last expression above is positive.

Combination of Inequalities

Combining the results of the last two theorems we have the following inequality.

$$S - K \leq C^a - P^a \leq S - Ke^{-rT}$$

Example

Example

The price is a security is currently \$36, the risk-free interest rate is 5.5% compounded continuously, and the strike price of a six-month American call option worth \$2.03 is \$37. The range of no arbitrage values of a six-month American put on the same security with the same strike price is

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$$\begin{aligned} S - K \leq C^a - P^a &\leq S - Ke^{-rT} \\ 36 - 37 \leq 2.03 - P^a &\leq 36 - 37e^{-0.055(6/12)} \end{aligned}$$

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A Surprising Equality

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Theorem

If C^a and C^e are the values of American and European call options respectively on the same underlying non-dividend-paying security with identical strike prices and expiry times, then

$$C^a = C^e.$$

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Theorem

If C^a and C^e are the values of American and European call options respectively on the same underlying non-dividend-paying security with identical strike prices and expiry times, then

$$C^a = C^e.$$

Remark: American calls on non-dividend-paying stocks are not exercised early.

Proof

Suppose that $C^a > C^e$.

- Sell the American call and buy a European call with the same strike price K, expiry date T, and underlying security. The net cash flow $C^a - C^e > 0$ would be invested at the risk-free rate r.
- If the owner of the American call chooses to exercise the option at some time $t \leq T$, sell short a share of the security for amount K and add the proceeds to the amount invested at the risk-free rate.
- At time T close out the short position in the security by exercising the European option. The amount due is

$$(C^a - C^e)e^{rT} + K(e^{r(T-t)} - 1) > 0.$$

- If the American option is not exercised, the European option can be allowed to expire and the amount due is

$$(C^a - C^e)e^{rT} > 0.$$

American Puts

Theorem

For a non-dividend-paying stock whose current price is S and for which the American put with a strike price of K and expiry T has a value of P^a, satisfies the inequality

$$(K - S)^+ \leq P^a < K.$$

Proof

- Suppose $P^a < K - S$.
  - Buy the put and the stock (cost $P^a + S$).
  - Immediately exercise the put and sell the stock for K.
  - Net transaction $K - P^a - S > 0$ (arbitrage).

Proof

- Suppose $P^a < K - S$.
  - Buy the put and the stock (cost $P^a + S$).
  - Immediately exercise the put and sell the stock for K.
  - Net transaction $K - P^a - S > 0$ (arbitrage).
- Suppose $P^a > K$.
  - Sell the put and invest proceeds at risk-free rate r. Amount due at time t is $P^a e^{rt}$.
  - If the owner of the put chooses to exercise it, buy the stock for K and sell it for $S(t)$. Net transaction $S(t) - K + P^a e^{rt} > S(t) + K(e^{rt} - 1) > 0$.
  - If the put expires unused, the profit is $P^a e^{rT} > 0$.

Example

Remark: in contrast to American calls, American puts on non-dividend-paying stocks will sometimes be exercised early.

Example

Consider a 12-month American put on a non-dividend paying stock currently worth \$15. If the risk-free interest rate is 3.25% per year and the strike price of the put is \$470, should the option be exercised early?

Solution

- We were not told the price of the put, but we know $P^a < K = 470$.
- If we exercise the put immediately, we gain $470 - 15 = 455$ and invest at the risk-free rate.
- In one year the amount due is $455e^{0.0325} = 470.03 > K > P^a$.

Thus the option should be exercised early.

American Calls

Theorem

For a non-dividend-paying stock whose current price is S and for which the American call with a strike price of K and expiry T has a value of C^a, satisfies the inequality

$$(S - Ke^{-rT})^+ \leq C^a < S.$$

Variables Determining Values of American Options (1 of 3)

Theorem

Suppose $T_1 < T_2$ *and*

- *let* $C^a(T_i)$ *be the value of an American call with expiry* T_i*, and*
- *let* $P^a(T_i)$ *be the value of an American put with expiry* T_i*,*

then

$$\begin{aligned} C^a(T_1) &\leq C^a(T_2) \\ P^a(T_1) &\leq P^a(T_2). \end{aligned}$$

Proof

Suppose $C^a(T_1) > C^a(T_2)$.

- Buy the option $C^a(T_2)$ and sell the option $C^a(T_1)$. Initial transaction,
$$C^a(T_1) > C^a(T_2) > 0$$
- If the owner of $C^a(T_1)$ chooses to exercise the option, we can exercise the option $C^a(T_2)$. Transaction cost,
$$(S(t) - K) - (S(t) - K) = 0.$$

Since we keep the initial transaction profit, arbitrage is present.

Variables Determining Values of American Options (2 of 3)

Theorem

Suppose $K_1 < K_2$ *and*

- *let* $C^a(K_i)$ *be the value of an American call with strike price* K_i*, and*
- *let* $P^a(K_i)$ *be the value of an American put with strike price* K_i*,*

then

$$\begin{aligned} C^a(K_2) &\leq C^a(K_1) \\ P^a(K_1) &\leq P^a(K_2) \\ C^a(K_1) - C^a(K_2) &\leq K_2 - K_1 \\ P^a(K_2) - P^a(K_1) &\leq K_2 - K_1. \end{aligned}$$

Variables Determining Values of American Options (3 of 3)

Theorem

Suppose $S_1 < S_2$ and

- *let $C^a(S_i)$ be the value of an American call written on a stock whose value is S_i, and*
- *let $P^a(S_i)$ be the value of an American put written on a stock whose value is S_i,*

then

$$\begin{aligned} C^a(S_1) &\leq C^a(S_2) \\ P^a(S_2) &\leq P^a(S_1) \\ C^a(S_2) - C^a(S_1) &\leq S_2 - S_1 \\ P^a(S_1) - P^a(S_2) &\leq S_2 - S_1. \end{aligned}$$

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$$\text{Define} \quad x_1 = \frac{S_1}{S(0)} \quad \text{and} \quad x_2 = \frac{S_2}{S(0)}.$$

- Sell x_1 options $C^a(S(0))$ where

$$x_1 C^a(S(0)) = C^a(S_1)$$

and buy x_2 options $C^a(S(0))$ where

$$x_2 C^a(S(0)) = C^a(S_2).$$

Initial transaction is

$$C^a(S_1) - C^a(S_2) > 0.$$

- If the owner of option $C^a(S_1)$ chooses to exercise it, option $C^a(S_2)$ is exercised as well. Transaction profit is

$$x_2(S(t) - K) - x_1(S(t) - K) = (x_2 - x_1)(S(t) - K) > 0.$$

Arbitrage is present.

Proof (3 of 3)

Suppose $P^a(S_1) - P^a(S_2) > S_2 - S_1$, this is equivalent to the inequality

$$P^a(S_1) + S_1 > P^a(S_2) + S_2.$$

- Buy x_2 put options $P^a(S(0))$, sell x_1 put options $P^a(S(0))$, and buy $x_2 - x_1$ shares of stock. Initial transaction,

$$\begin{aligned} & x_1 P^a(S(0)) - x_2 P^a(S(0)) - (x_2 - x_1)S(0) \\ = \quad & P^a(S_1) - P^a(S_2) - (S_2 - S_1) > 0. \end{aligned}$$

- If the owner of put $P^a(S_1)$ chooses to exercise the option, we exercise put $P^a(S_2)$ and sell our $x_2 - x_1$ shares of stock.

$$(x_2 - x_1)S(t) + x_2(K - S(t)) - x_1(K - S(t)) = (x_2 - x_1)K > 0$$

Arbitrage is present.

Binomial Pricing of American Puts

Assumptions:

- Strike price of the American put is K,
- Expiry date of the American put is $T > 0$,
- Price of the security at time t with $0 \leq t \leq T$ is $S(t)$,
- Continuously compounded risk-free interest rate is r, and
- Price of the security follows a geometric Brownian motion with variance σ^2.

Definitions

u: factor by which the stock price may increase during a time step.

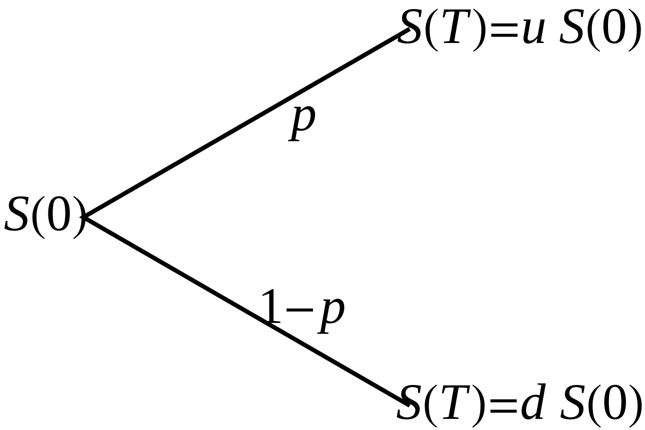
$$u = e^{\sigma\sqrt{\Delta t}} > 1$$

d: factor by which the stock price may decrease during a time step.

$$0 < d = e^{-\sigma\sqrt{\Delta t}} < 1$$

p: probability of an increase in stock price during a time step.

$$0 < p = \frac{1}{2}\left(1 + \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)\sqrt{\Delta t}\right) < 1$$


S(T)=u S(0)
p
S(0)
1−p
S(T)=d S(0)

Intrinsic Value

Observation: an American put is always worth at least as much as the payoff generated by immediate exercise.

Definition

The **intrinsic value** at time t of an American put is the quantity $(K - S(t))^+$.

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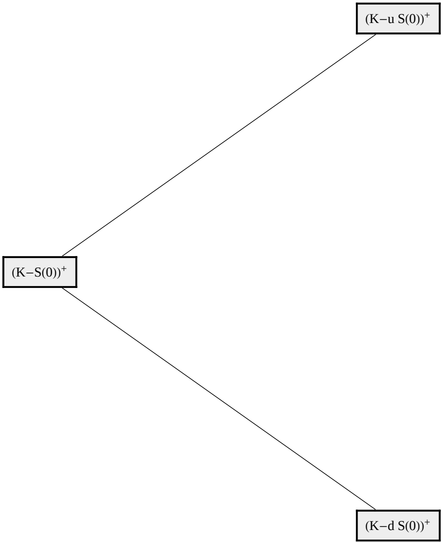
The **intrinsic value** at time t of an American put is the quantity $(K - S(t))^+$.

The value of an American put is the greater of its intrinsic value and the present value of its expected intrinsic value at the next time step.

One-Step Illustration (1 of 2)



One-Step Illustration (2 of 2)

At expiry the American put is worth

$$P^a(T) = \begin{cases} (K - uS(0))^+ & \text{with probability } p, \\ (K - dS(0))^+ & \text{with probability } 1 - p. \end{cases}$$

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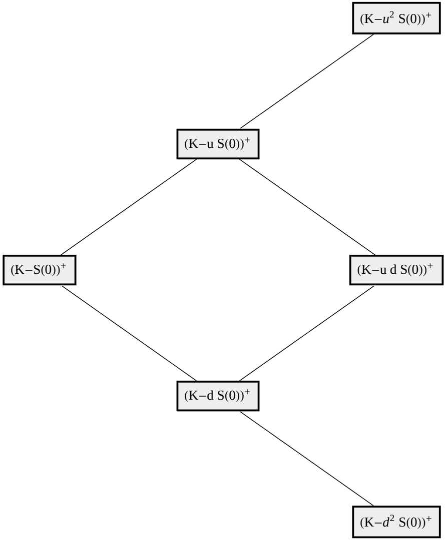
At $t = 0$ the American put is worth

$$\begin{aligned} P^a(0) &= \max\{(K - S(0))^+, \\ &\qquad e^{-rT}\left[p(K - uS(0))^+ + (1 - p)(K - dS(0))^+\right]\right\} \\ &= \max\left\{(K - S(0))^+, e^{-rT}\mathrm{E}\left[(K - S(T))^+\right]\right\} \\ &= \max\left\{(K - S(0))^+, e^{-rT}\mathrm{E}\left[P^a(T)\right]\right\}. \end{aligned}$$

Two-Step Ilustration (1 of 2)



Two-Step Ilustration (2 of 2)

At $t = T/2$, if the put has not been exercised already, an investor will exercise it, if the option is worth more than the present value of the expected value at $t = T$.

$$P^a(T/2) = \max\left\{(K - S(T/2))^+, e^{-rT/2}\mathrm{E}\left[P^a(T)\right]\right\}$$

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Using the same logic, the value of the put at $t = 0$ is the larger of the intrisic value at $t = 0$ and the present value of the expected value at $t = T/2$.

$$P^a(0) = \max\left\{(K - S(0))^+, e^{-rT/2}\mathrm{E}\left[P^a(T/2)\right]\right\}$$

Example

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Suppose the current price of a security is \$32, the risk-free interest rate is 10% compounded continuously, and the volatility of Brownian motion for the security is 20%. Find the price of a two-month American put with a strike price of \$34 on the security.

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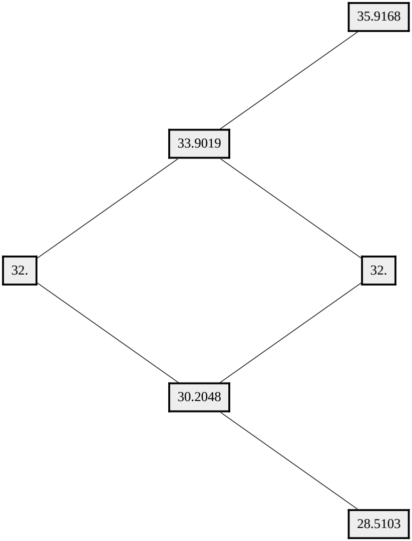
Suppose the current price of a security is \$32, the risk-free interest rate is 10% compounded continuously, and the volatility of Brownian motion for the security is 20%. Find the price of a two-month American put with a strike price of \$34 on the security.

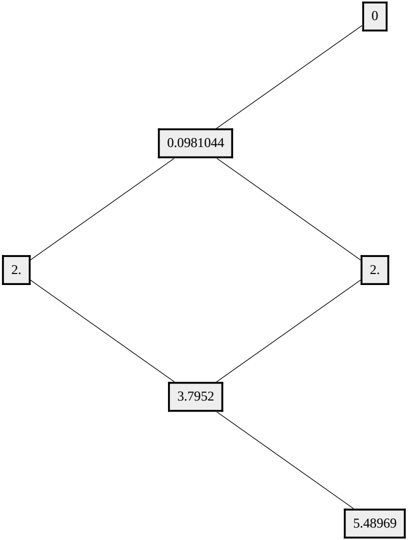
We will set $\Delta t = 1/12$, then

$$\begin{aligned} u &\approx 1.0594 \\ d &\approx 0.9439 \\ p &\approx 0.5574. \end{aligned}$$

Stock Price Lattice



Intrinsic Value Lattice



Pricing the Put at $t = 1/12$

If $S(1/12) = 33.9019$ then

$$\begin{aligned} P^a(1/12) &= \max\left\{(34 - 33.9019)^+, \right. \\ &\quad e^{-0.10/12}(0.5574(34 - 35.9168)^+ \\ &\quad\quad \left. + (1 - 0.5574)(34 - 32)^+)\right\} \\ &= 0.8779. \end{aligned}$$

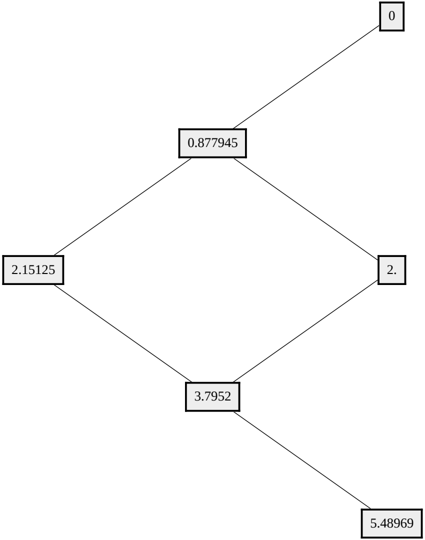
If $S(1/12) = 30.2048$ then

$$\begin{aligned} P^a(1/12) &= \max\left\{(34 - 30.2048)^+, \right. \\ &\quad e^{-0.10/12}(0.5574(34 - 32)^+ \\ &\quad\quad \left. + (1 - 0.5574)(34 - 28.5103)^+)\right\} \\ &= 3.7942. \end{aligned}$$

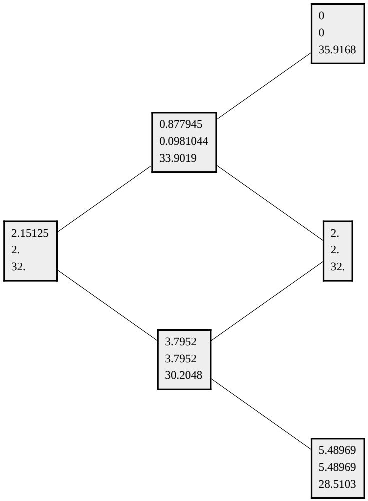
Pricing the Put at $t = 0$

$$\begin{aligned} P^a(0) &= \max\left\{(34-32)^+, \right. \\ &\quad \left. e^{-0.10/12}[(0.5574)(0.8779)+(0.4426)(3.7952)]\right\} \\ &= 2.1513. \end{aligned}$$

American Put Lattice



Summary: $\begin{bmatrix} P^a(t) \\ (K - S(t))^+ \\ S(t) \end{bmatrix}$



General Pricing Framework

Using a recursive procedure, the value of an American option, for example a put, is given by

$$
\begin{aligned}
P^a(T) &= (K - S(T))^+ \\
P^a((n-1)\Delta t) &= \max\left\{(K - S((n-1)\Delta t))^+, e^{-r\Delta t}\mathrm{E}\left[P^a(T)\right]\right\} \\
P^a((n-2)\Delta t) &= \max\left\{(K - S((n-2)\Delta t))^+,\right. \\
&\qquad \left. e^{-r\Delta t}\mathrm{E}\left[P^a((n-1)\Delta T)\right]\right\} \\
&\vdots \\
P^a(0) &= \max\left\{(K - S(0))^+, e^{-r\Delta t}\mathrm{E}\left[P^a(\Delta T)\right]\right\}.
\end{aligned}
$$

If a stock pays a dividend during the life of an American call option, it may be advantageous to exercise the call early so as to collect the dividend.

Example

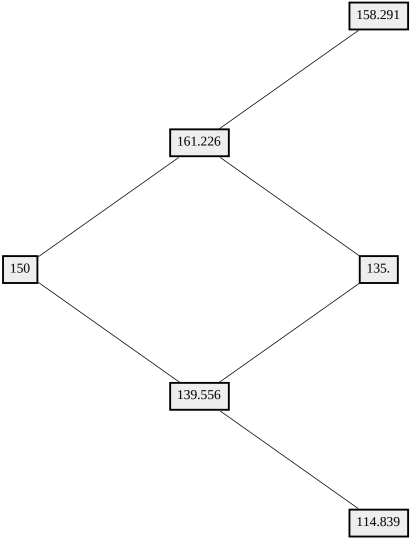
Suppose a stock is currently worth \$150 and has a volatility of 25% per year. The stock will pay a dividend of \$15 in two months. The risk-free interest rate is 3.25%. Find the prices of two-month European and American call options on the stock with strikes prices of \$150.

Solution (1 of 3)

If $\Delta t = 1/12$, then

$$
\begin{aligned}
u = e^{\sigma\sqrt{\Delta t}} &\approx 1.07484 \\
d = e^{-\sigma\sqrt{\Delta t}} &\approx 0.930374 \\
p = \frac{1}{2}\left(1 + \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)\sqrt{\Delta t}\right) &\approx 0.500722.
\end{aligned}
$$

Solution (2 of 3) Stock Prices



Solution (3 of 3) Call Prices