Discrete Random Variables

An Undergraduate Introduction to Financial Mathematics

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Introduction

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We must understand:

- random variables,
- elementary rules of probability,
- expected value,
- variance.

Basic Definitions

Definition

- An **experiment** is any activity that generates an observable outcome.
- An **event** is an outcome or set of outcomes with a specified property (generally denoted with letters A, B, $\ldots$).
- The **probability** of an event is a real number measuring the likelihood of the occurrence of the event (generally denoted $\mathbb{P}(A)$, $\mathbb{P}(B)$, $\ldots$).

Discrete Events

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3. Drawing a card from a standard deck

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Later, we may think of discrete outcomes as the results of experiments with a finite or at most countable number of outcomes.

Properties of Probability

If A is an event,

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To determine the $\mathbb{P}(A)$ we can:

- take the **empirical approach** and conduct (or at least simulate) the experiment N times, count the number of times x that event A occurred, and estimate $\mathbb{P}(A) = x/N$.
- take the **classical approach** and determine the number of outcomes of the experiment (call this number M), assume the outcomes are equally likely, and determine the number of outcomes y among the M in which event A occurs. Then $\mathbb{P}(A) = y/M$.

Determining the Size of the Sample Space

Determine the number of outcomes of the experiment of tossing a pair of dice twice.

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$$M = (36)^2 = 1296$$

Determining the Classical Probability

You and a friend each toss a fair coin. You win if your coin matches your friend's coin.

1. Determine the sample space of the experiment.
2. What is the probability that you win?

Notation: $\mathbb{P}(A \vee B)$ denotes the probability that event A or B occurs.

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Definition

Two events are **mutually exclusive** if they cannot occur together.

Addition Rule (2 of 2)

Theorem (Addition Rule)

$\mathbb{P}(A \vee B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \wedge B)$ *where* $\mathbb{P}(A \wedge B)$ *is the probability that events A and B occur together.*

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Corollary

If A and B are mutually exclusive then $\mathbb{P}(A \vee B) = \mathbb{P}(A) + \mathbb{P}(B)$.

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Corollary

If A and B are mutually exclusive then $\mathbb{P}(A \vee B) = \mathbb{P}(A) + \mathbb{P}(B)$.

Remark: since A and B are mutually exclusive, $A \wedge B$ is an impossible event.

Example

Find the probability that when a pair of fair dice are thrown the total of the dice is less than 6 or odd.

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Solution:

$$\begin{aligned}
\mathbb{P}((X < 6) \vee (X \text{ odd})) &= \mathbb{P}(X < 6) + \mathbb{P}(X \text{ odd}) \\
&\quad - \mathbb{P}((X < 6) \wedge (X \text{ odd})) \\
&= \frac{10}{36} + \frac{18}{36} - \frac{6}{36} \\
&= \frac{11}{18}
\end{aligned}$$

Monty Hall Problem

A game show host hides a prize behind one of three doors. A contestant must guess which door hides the prize. First, the contestant announces the door they have chosen. The host will then open one of the two doors, not chosen, in order to reveal the prize is not behind it. The host then tells the contestant they may keep their original choice or switch to the other unopened door. Should the contestant switch doors?

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Remark: you may wish to try a simulator for this game.

Sampling with Replacement

There are three red socks and six black socks in a drawer. What is the probability that in a sample of size 2 taken with replacement, that

1. the sample contains a red sock?
2. the sample contains exactly one red sock?
3. the sample contains two red socks?

Conditional Probability

Definition

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What is the contestant's probability of winning if they do not switch doors? Should they switch?

Multiplication Rule

Theorem (Multiplication Rule)

For events A and B, the probability of A and B occurring is

$$\mathbb{P}(A \wedge B) = \mathbb{P}(A)\,\mathbb{P}(B|A),$$

or equivalently

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A \wedge B)}{\mathbb{P}(A)}.$$

provided $\mathbb{P}(A) > 0$.

Example

There are two drawers (labeled α and β). Drawer α contains only red socks. Drawer β contains equals numbers of red socks and black socks. A fair coin is tossed to determine a drawer to select from and a red sock is drawn.

What is the probability the drawer contains only red socks?

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What is the probability the drawer contains only red socks?

$$\mathbb{P}\left(\alpha \,|\, \text{red}\right) = \frac{\mathbb{P}\left(\alpha \wedge \text{red}\right)}{\mathbb{P}\left(\text{red}\right)} = \frac{\frac{1}{2}}{\frac{1}{2} + \frac{1}{4}} = \frac{2}{3}$$

Example: Roulette (1 of 2)

One type of roulette wheel, known as the American type, has 38 potential outcomes represented by the integers 1 through 36 and two special outcomes 0 and 00. The positive integers are placed on alternating red and black backgrounds while 0 and 00 are on green backgrounds.

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Question: What is the probability that the outcome is less than 10 and more than 3 given that the outcome is an even number? (0 and 00 are considered even numbers.)

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Solution:

$$\begin{aligned}
\mathbb{P}\left((3 < X < 10) \,|\, (X \text{ even})\right) &= \frac{\mathbb{P}\left((3 < X < 10) \wedge (X \text{ even})\right)}{\mathbb{P}\left(X \text{ even}\right)} \\
&= \frac{3/38}{20/38} \\
&= \frac{3}{20}
\end{aligned}$$

Independent Events

Definition

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Corollary

If events A and B are independent then

$$\mathbb{P}(A \wedge B) = \mathbb{P}(A)\,\mathbb{P}(B).$$

Example

Question: what is the probability of two red outcomes on successive spins of the roulette wheel?

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Answer: since the outcomes of spins of the roulette wheel are independent, then

$$\begin{aligned}\mathbb{P}(\text{1st spin red} \wedge \text{2nd spin red}) &= \mathbb{P}(\text{1st spin red})\,\mathbb{P}(\text{2nd spin red}) \\ &= \left(\frac{18}{38}\right)\left(\frac{18}{38}\right) \\ &= \frac{81}{361} \\ &\approx 0.224377.\end{aligned}$$

Random Variables

Definition

A **random variable** (abbreviated RV) is a function $X : S \to \mathbb{R}$ where S is the space of outcomes of an experiment.

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$$X \in \{0, 1, \ldots, N\}$$

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$$X \in \{0, 1, \ldots, N\}$$

Remark: the value of X is unknown until the experiment is conducted.

Probability Functions

Definition

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If $S = \{x_1, x_2, \ldots, x_N\}$ and f is a probability function then:

1. $0 \leq f(x_i) \leq 1$ for $i = 1, 2, \ldots, N$, and
2. $1 = \sum_{i=1}^{N} f(x_i)$.

Definition

A **Bernoulli** random variable can be thought of as having sample space $S = \{0, 1\}$ and probability function

$$f(x) = \begin{cases} p & \text{if } x = 1, \\ 1 - p & \text{if } x = 0. \end{cases}$$

The 0 outcome is often termed “failure” while the 1 outcome is called “success”.

Bernoulli and Binomial Random Variables (1 of 2)

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Definition

A **binomial** random variable is the number of successes out of n independent Bernoulli experiments. The number of trials n is fixed and the Bernoulli probabilities remain fixed between trials.

Binomial Random Variables (2 of 2)

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ways to choose x items out of a collection of n items.

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Thus $\mathbb{P}(x)$ is given by

$$\mathbb{P}(x) = \binom{n}{x} p^x (1-p)^{n-x} = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}.$$

Example (1 of 2)

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Assuming the genders of the children are independent and equally likely to be male or female, the probability of the desired outcome is the binomial probability,

$$\mathbb{P}\,(\text{exactly 3 of 4 children male}) = \binom{4}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right) = \frac{1}{4}.$$

Example (2 of 2)

The probability that a computer memory chip is defective is 0.02. A SIMM (single in-line memory module) contains 16 chips for data storage and a 17th chip for error correction. The SIMM can operate correctly if one chip is defective, but not if two or more are defective. What is the probability that the SIMM functions correctly?

Example (2 of 2)

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The SIMM will function correctly if there is at most one malfunctioning memory chip.

$$\begin{aligned}
&\mathbb{P}\,(\text{no bad chips} \vee \text{one bad chip}) \\
&\quad= \mathbb{P}\,(\text{no bad chips}) + \mathbb{P}\,(\text{one bad chip}) \\
&\quad= (1-0.02)^{17} + \binom{17}{1}(0.02)(1-0.02)^{16} \\
&\quad\approx 0.955413
\end{aligned}$$

Expected Value of a Random Variable

Definition

If X is a discrete random variable with probability $\mathbb{P}(X)$ then the **expected value of** X is denoted $\mathbb{E}[X]$ and defined as

$$\mathbb{E}[X] = \sum_{X} (X \cdot \mathbb{P}(X)).$$

It is understood that the summation is taken over all values that X may assume.

Example

What is the expected value of the number of female children in a family of 5 children?

Example

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If X represents the number of female children in a family with five children then

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{X} (X \cdot \mathbb{P}(X)) \\
&= \sum_{x=0}^{5} \left(x \cdot \binom{5}{x} \left(\frac{1}{2}\right)^{x} \left(\frac{1}{2}\right)^{5-x} \right) \\
&= \frac{1}{32} \sum_{x=0}^{5} \frac{(5!)x}{(x!)(5-x)!} \\
&= 2.5
\end{aligned}$$

Expected Value of a Function of a Random Variable

Definition

If F is a function of the random variable X, then the **expected value of** F is

$$\mathbb{E}\left[F(X)\right] = \sum_{X} F(X)\mathbb{P}\left(X\right).$$

Example

What is the expected value of the square of the number of female children in a family of 5 children?

Example

What is the expected value of the square of the number of female children in a family of 5 children?

If X represents the number of female children in a family with five children then

$$\begin{aligned}
\mathbb{E}\left[X^2\right] &= \sum_X \left(X^2 \cdot \mathbb{P}\left(X\right)\right) \\
&= \sum_{x=0}^{5} \left(x^2 \cdot \binom{5}{x} \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{5-x}\right) \\
&= 7.5
\end{aligned}$$

Linearity of Expected Value

Theorem

If X is a random variable and a is a constant, then $\mathbb{E}[aX] = a\mathbb{E}[X]$.

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If X is a random variable and a is a constant, then $\mathbb{E}\left[a\,X\right] = a\,\mathbb{E}\left[X\right]$.

Proof.

$$\mathbb{E}\left[a\,X\right] = \sum_{X} \left((aX) \cdot \mathbb{P}\left(X\right)\right) = a \sum_{X} \left(X \cdot \mathbb{P}\left(X\right)\right) = a\,\mathbb{E}\left[X\right].$$

□

Joint Probability Function

Definition

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A joint probability has the properties:

1. $0 \leq \mathbb{P}(X, Y) \leq 1$,
2. $\displaystyle\sum_X \sum_Y \mathbb{P}(X, Y) = \sum_Y \sum_X \mathbb{P}(X, Y) = 1$.

Marginal Probability

Definition

If the joint probability of X and Y is $\mathbb{P}(X, Y)$ then the quantity

$$\sum_{Y} \mathbb{P}(X, Y)$$

is called the **marginal probability of** X. We will then write $\mathbb{P}(X) = \sum_{Y} \mathbb{P}(X, Y)$.

We may define a **marginal probability for** Y similarly.

Expected Value of a Sum (1 of 2)

Theorem

If $X_1, X_2, \ldots, X_k$ are random variables then

$$\mathbb{E}\left[X_1 + X_2 + \cdots + X_k\right] = \mathbb{E}\left[X_1\right] + \mathbb{E}\left[X_2\right] + \cdots + \mathbb{E}\left[X_k\right].$$

Expected Value of a Sum (2 of 2)

Proof.

If $k = 1$ then the proposition is certainly true. If $k = 2$ then

$$\begin{aligned}
\mathbb{E}[X_1 + X_2] &= \sum_{X_1}\sum_{X_2}((X_1 + X_2)\mathbb{P}(X_1, X_2)) \\
&= \sum_{X_1}\sum_{X_2} X_1\mathbb{P}(X_1, X_2) + \sum_{X_2}\sum_{X_1} X_2\mathbb{P}(X_1, X_2) \\
&= \sum_{X_1} X_1 \sum_{X_2}\mathbb{P}(X_1, X_2) + \sum_{X_2} X_2 \sum_{X_1}\mathbb{P}(X_1, X_2) \\
&= \sum_{X_1} X_1\mathbb{P}(X_1) + \sum_{X_2} X_2\mathbb{P}(X_2) \\
&= \mathbb{E}[X_1] + \mathbb{E}[X_2].
\end{aligned}$$

The general case then holds by induction on k. □

Expected Value of a Sum of Functions

Corollary

Let $X_1, X_2, \ldots, X_k$ be random variables and let F_i be a function of X_i for $i = 1, 2, \ldots, k$ then

$$\mathbb{E}\left[F_1(X_1) + \cdots + F_k(X_k)\right] = \mathbb{E}\left[F_1(X_1)\right] + \cdots + \mathbb{E}\left[F_k(X_k)\right].$$

Expected Value of Binomial Random Variable

If X_i is a Bernoulli random variable then

$$\mathbb{E}[X_i] = (1)(p) + (0)(1-p) = p.$$

Expected Value of Binomial Random Variable

If X_i is a Bernoulli random variable then

$$\mathbb{E}\left[X_i\right] = (1)(p) + (0)(1 - p) = p.$$

If the binomial random variable $X = X_1 + \cdots + X_n$ where each X_i is a Bernoulli random variable, then

$$\mathbb{E}\left[X\right] = \mathbb{E}\left[X_1\right] + \cdots + \mathbb{E}\left[X_n\right] = n\,p.$$

Independent Random Variables (1 of 2)

If X and Y are *independent* random variables then

$$\mathbb{P}(X, Y) = \mathbb{P}(X)\,\mathbb{P}(Y)$$

where $\mathbb{P}(X)$ is the probability of X and $\mathbb{P}(Y)$ is the probability of Y.

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Theorem

Let $X_1, X_2, \ldots, X_k$ be pairwise independent random variables, then

$$\mathbb{E}[X_1 X_2 \cdots X_k] = \mathbb{E}[X_1]\,\mathbb{E}[X_2] \cdots \mathbb{E}[X_k].$$

Proof.

When $k = 1$ the theorem is true. Now let X_1 and X_2 be independent random variables with joint probability $\mathbb{P}(X_1, X_2)$.

$$\begin{aligned} \mathbb{E}[X_1 X_2] &= \sum_{X_1, X_2} X_1 X_2 \mathbb{P}(X_1, X_2) \\ &= \sum_{X_1} \sum_{X_2} X_1 X_2 \mathbb{P}(X_1) \mathbb{P}(X_2) \\ &= \sum_{X_1} X_1 \mathbb{P}(X_1) \sum_{X_2} X_2 \mathbb{P}(X_2) \\ &= \mathbb{E}[X_1] \mathbb{E}[X_2] \end{aligned}$$

The general case holds by induction on k. □

Definition

If X is a random variable, the **variance** of X is denoted $\mathbb{V}(X)$ and

$$\mathbb{V}(X) = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right].$$

The **standard deviation** of X is denoted $\sigma(X) = \sqrt{\mathbb{V}(X)}$.

Variance (2 of 2)

Theorem

Let X be a random variable, then the variance of X is

$$\mathbb{V}\left(X\right) = \mathbb{E}\left[X^2\right] - \mathbb{E}\left[X\right]^2.$$

Theorem

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$$\mathbb{V}(X) = \mathbb{E}\left[X^2\right] - \mathbb{E}[X]^2.$$

Proof.

$$\begin{aligned}
\mathbb{V}(X) &= \mathbb{E}\left[(X - \mathbb{E}[X])^2\right] \\
&= \mathbb{E}\left[X^2\right] - \mathbb{E}[2X\mathbb{E}[X]] + \mathbb{E}\left[\mathbb{E}[X]^2\right] \\
&= \mathbb{E}\left[X^2\right] - 2\mathbb{E}[X]\,\mathbb{E}[X] + \mathbb{E}[X]^2 \\
&= \mathbb{E}\left[X^2\right] - \mathbb{E}[X]^2.
\end{aligned}$$

$\square$

Example (1 of 2)

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If X represents the number of female children in a family of five children, then

$$\mathbb{V}(X) = \mathbb{E}\left[X^2\right] - (\mathbb{E}[X])^2 = 7.5 - (2.5)^2 = 1.25 .$$

Example (2 of 2)

Find the variance of a Bernoulli random variable for which the probability of success is p.

Example (2 of 2)

Find the variance of a Bernoulli random variable for which the probability of success is p.

If X represents the outcome of a Bernoulli trial, then

$$\begin{aligned}
\mathbb{V}(X) &= \mathbb{E}\left[X^2\right] - (\mathbb{E}[X])^2 \\
&= \left[1^2(p) + 0^2(1-p)\right] - p^2 \\
&= p - p^2 \\
&= p(1-p).
\end{aligned}$$

Variance of a Sum (1 of 3)

Theorem

Let $X_1, X_2, \ldots, X_k$ be pairwise independent random variables, then

$$\mathbb{V}(X_1 + X_2 + \cdots + X_k) = \mathbb{V}(X_1) + \mathbb{V}(X_2) + \cdots + \mathbb{V}(X_k).$$

If $k = 1$ then the result is trivially true. Take the case when $k = 2$. By the definition of variance,

$$\begin{aligned}
\mathbb{V}(X_1 + X_2) &= \mathbb{E}\left[((X_1 + X_2) - \mathbb{E}[X_1 + X_2])^2\right] \\
&= \mathbb{E}\left[((X_1 - \mathbb{E}[X_1]) + (X_2 - \mathbb{E}[X_2]))^2\right] \\
&= \mathbb{E}\left[(X_1 - \mathbb{E}[X_1])^2\right] + \mathbb{E}\left[(X_2 - \mathbb{E}[X_2])^2\right] + \\
&\quad 2\mathbb{E}[(X_1 - \mathbb{E}[X_1])(X_2 - \mathbb{E}[X_2])] \\
&= \mathbb{V}(X_1) + \mathbb{V}(X_2) \\
&\quad + 2\mathbb{E}[(X_1 - \mathbb{E}[X_1])(X_2 - \mathbb{E}[X_2])]
\end{aligned}$$

Variance of a Sum (2 of 3)

Since we are assuming that random variables X_1 and X_2 are independent, then

$$\begin{aligned}\mathbb{E}\left[(X_1 - \mathbb{E}\left[X_1\right])(X_2 - \mathbb{E}\left[X_2\right])\right] &= \mathbb{E}\left[X_1 - \mathbb{E}\left[X_1\right]\right]\mathbb{E}\left[X_2 - \mathbb{E}\left[X_2\right]\right] \\ &= (\mathbb{E}\left[X_1\right] - \mathbb{E}\left[X_1\right])(\mathbb{E}\left[X_2\right] - \mathbb{E}\left[X_2\right]) \\ &= 0,\end{aligned}$$

and thus

$$\mathbb{V}\left(X_1 + X_2\right) = \mathbb{V}\left(X_1\right) + \mathbb{V}\left(X_2\right).$$

The result can be extended to any finite value of k by induction.

Example

Find the variance of a Binomial random variable with n trials when the probability of success on a single trial is p.

Example

Find the variance of a Binomial random variable with n trials when the probability of success on a single trial is p.

Each trial of a binomial experiment is a Bernoulli experiment with probability of success p. Since the n trials of a binomial experiment are independent

$$\mathbb{V}(X) = n\,p(1-p).$$

Variance of a Product (1 of 2)

Theorem

Let $X_1, X_2, \ldots, X_k$ be pairwise independent random variables, then

$$\begin{aligned} \mathbb{V}(X_1 X_2 \cdots X_k) &= \mathbb{E}\left[X_1^2\right] \mathbb{E}\left[X_2^2\right] \cdots \mathbb{E}\left[X_k^2\right] \\ &\quad - \left(\mathbb{E}[X_1]\,\mathbb{E}[X_2] \cdots \mathbb{E}[X_k]\right)^2 . \end{aligned}$$

Variance of a Product (2 of 2)

Proof.

$$\begin{aligned}
\mathbb{V}(X_1 X_2 \cdots X_k) &= \mathbb{E}\left[(X_1 X_2 \cdots X_k)^2\right] - (\mathbb{E}[X_1 X_2 \cdots X_k])^2 \\
&= \mathbb{E}\left[X_1^2 X_2^2 \cdots X_k^2\right] - (\mathbb{E}[X_1]\,\mathbb{E}[X_2] \cdots \mathbb{E}[X_k])^2 \\
&= \mathbb{E}\left[X_1^2\right]\mathbb{E}\left[X_2^2\right] \cdots \mathbb{E}\left[X_k^2\right] \\
&\quad - (\mathbb{E}[X_1]\,\mathbb{E}[X_2] \cdots \mathbb{E}[X_k])^2
\end{aligned}$$

□

Credits

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