Extensions to the Black-Scholes Equation

An Undergraduate Introduction to Financial Mathematics

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2014

Dividends

- We have versions of the Put-Call Parity formula which include the effects of dividends:

$$
\begin{aligned}
P_e + Se^{-\delta T} &= C_e + Ke^{-rT} \qquad \text{(continuous)} \\
P_e + S(0) - \delta \sum_{i=1}^{n} S(t_i^-)e^{-rt_i} &= C_e + Ke^{-rT} \qquad \text{(discrete)}
\end{aligned}
$$

- We do not have pricing formulas for the options themselves. We explore modifications and extensions to the Black-Scholes partial differential equation and its solution in this lesson.

Basic Problem for the European Call

The non-dividend-paying stock is assumed to obey the stochastic process

$$dS = \mu S\,dt + \sigma S\,dW(t)$$

and the European call solves the initial boundary value problem:

$$\begin{aligned}
rF &= F_t + rSF_S + \frac{1}{2}\sigma^2 S^2 F_{SS} \quad \text{for } (S,t) \text{ in } [0,\infty) \times [0,T], \\
F(S,T) &= (S(T) - K)^+ \quad \text{for } S > 0, \\
F(0,t) &= 0 \quad \text{for } 0 \leq t < T, \\
F(S,t) &= S - Ke^{-r(T-t)} \quad \text{as } S \to \infty.
\end{aligned}$$

Stock Pays Continuous Dividends

Assumption: the stock pays dividends at a continuous rate proportional to the value of the stock

- What is a suitable expression for the dividend yield (dividend paid per unit time)?
- How much dividend is paid in a short time interval dt?
- What stochastic differential equation would the value of the stock paying a continuous proportional dividend obey?

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$$\text{dividend paid} = \delta S\, dt$$

- What stochastic differential equation would the value of the stock paying a continuous proportional dividend obey?

$$dS = (\mu - \delta) S\, dt + \sigma S\, dW(t)$$

Suppose $F(S,t)$ is the value of a European call option on the stock paying a continuous dividend, F obeys the following stochastic differential equation:

$$dF = \left((\mu - \delta) S F_S + \frac{1}{2}\sigma^2 S^2 F_{SS} + F_t \right) dt + \sigma S F_S \, dW(t).$$

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As before, we wish to eliminate the random part of this equation by creating a portfolio of a long position in the call option and a short position in Δ shares of the stock.

$$P = F - (\Delta) S$$

Change in Portfolio Value

One share of stock pays $\delta S\,dt$ in dividends during a time interval of length dt, thus Δ shares of stock pays

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The portfolio changes in value

$$\begin{aligned}
dP &= d(F-(\Delta)S) - \delta(\Delta)S\,dt \\
&= dF - (\Delta)dS - \delta(\Delta)S\,dt \\
&= \left((\mu-\delta)SF_S + \frac{1}{2}\sigma^2 S^2 F_{SS} + F_t\right)dt + \sigma SF_S\,dW(t) \\
&\quad - (\Delta)\left((\mu-\delta)S\,dt + \sigma S\,dW(t)\right) - \delta(\Delta)S\,dt \\
&= \left((\mu-\delta)S(F_S-\Delta) + \frac{1}{2}\sigma^2 S^2 F_{SS} + F_t - \delta(\Delta)S\right)dt \\
&\quad + \sigma S(F_S-\Delta)\,dW(t).
\end{aligned}$$

Eliminating Randomness

Choose $\Delta = F_S$ and the portfolio obeys the stochastic differential equation:

$$dP = \left(\frac{1}{2}\sigma^2 S^2 F_{SS} + F_t - \delta S F_S \right) dt.$$

In the absence of arbitrage the change in the value of the portfolio should be the same as the interest earned by a equivalent amount of cash.

$$dP = r(F - (\Delta)S)\, dt$$

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$$dP = r(F - (\Delta)S)\, dt$$

Thus the Black-Scholes partial differential equation for the stock paying continuous dividends becomes

$$rF = F_t + \frac{1}{2}\sigma^2 S^2 F_{SS} + (r - \delta) S F_S.$$

Similarities with Non-Dividend-Paying Stocks

- Payoff of the call option at expiry: $F(S,T) = (S(T) - K)^+$.
- Boundary condition at $S = 0$ is $F(0,t) = 0$.
- Boundary condition as $S \to \infty$:

$$\begin{aligned} F(S,t) &= P_e + Se^{-\delta(T-t)} - Ke^{-r(T-t)} \\ \lim_{S\to\infty} F(S,t) &= \lim_{S\to\infty} \left(P_e + Se^{-\delta(T-t)} - Ke^{-r(T-t)} \right) \\ &= Se^{-\delta(T-t)} - Ke^{-r(T-t)}. \end{aligned}$$

Change of Variables

Define the function $G(S,t) = e^{\delta(T-t)}F(S,t)$, then

$$\begin{aligned}
G(S,T) &= e^{\delta(T-T)}F(S,T) = (S(T)-K)^+ \\
G(0,t) &= e^{\delta(T-t)}F(0,t) = 0 \\
\lim_{S\to\infty} G(S,t) &= e^{\delta(T-t)}\left(Se^{-\delta(T-t)} - Ke^{-r(T-t)}\right) \\
&= S - e^{-(r-\delta)(T-t)} \\
F_S &= e^{-\delta(T-t)}G_S \\
F_{SS} &= e^{-\delta(T-t)}G_{SS} \\
F_t &= e^{-\delta(T-t)}(\delta G + G_t).
\end{aligned}$$

Substitute these expressions into the partial differential equation, boundary conditions, and the final condition for the European call option on the stock paying continuous dividends.

Initial Boundary Value Problem

$$
\begin{aligned}
(r-\delta)G &= G_t + \frac{1}{2}\sigma^2 S^2 G_{SS} + (r-\delta) S G_S \\
G(S,T) &= (S(T)-K)^+ \\
G(0,t) &= 0 \\
\lim_{S\to\infty} G(S,t) &= S - e^{-(r-\delta)(T-t)}
\end{aligned}
$$

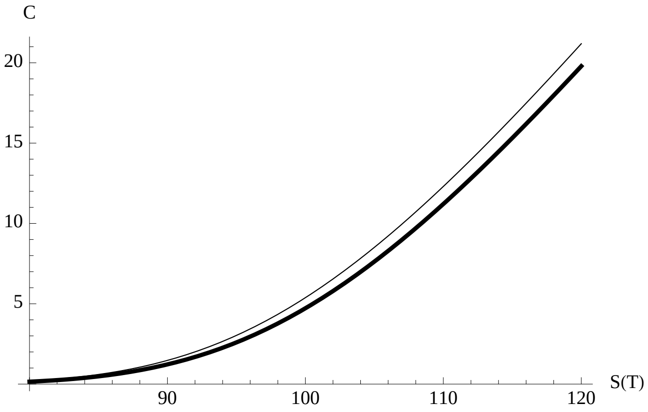
Remark: this is exactly the same initial boundary value problem we have already solved except r has been replaced by $r-\delta$.

Black-Scholes Option Pricing Formulas

For a stock paying a continuous, proportional dividend at rate δ the value of a European options are given by the formulas

$$
\begin{aligned}
w &= \frac{\ln(S/K) + (r - \delta + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}} \\
C_e^\delta &= e^{-\delta(T-t)} S\Phi\left(w\right) - Ke^{-r(T-t)}\Phi\left(w - \sigma\sqrt{T - t}\right) \\
P_e^\delta &= Ke^{-r(T-t)}\Phi\left(\sigma\sqrt{T - t} - w\right) - e^{-\delta(T-t)} S\Phi\left(-w\right)
\end{aligned}
$$

Comparison



The lighter curve represents a call option on a stock paying no dividends, while the heavier curve represents an otherwise identical call option paying a continuous dividend.

Example: Call Option

Suppose the current price of a security is \$62 per share. The continuously compounded interest rate is 10% per year. The volatility of the price of the security is $\sigma = 20\%$ per year. The stock pays dividends continuously at a rate of $\delta = 3\%$ per year. Find the cost of a five-month European call option with a strike price of \$60 per share.

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$$T = \frac{5}{12}, \quad t = 0, \quad r = 0.10, \quad \sigma = 0.20,$$
$$S = 62, \quad K = 60, \quad \delta = 0.03$$

Using the formula for w and C_e^δ we have

$$\begin{aligned} w &\approx 0.544463 \\ C_e^\delta &\approx 5.24 \end{aligned}$$

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Without the dividend we calculate $C_e \approx 5.80$.

Example: Put Option

Suppose the current price of a security is \$97 per share. The stock pays a continuous dividend at a yield of 6.5% per year. The continuously compounded interest rate is 8% per year. The volatility of the price of the security is $\sigma = 45\%$ per year. Find the cost of a three-month European put option with a strike price of \$95 per share.

Example: Put Option

Suppose the current price of a security is \$97 per share. The stock pays a continuous dividend at a yield of 6.5% per year. The continuously compounded interest rate is 8% per year. The volatility of the price of the security is $\sigma = 45\%$ per year. Find the cost of a three-month European put option with a strike price of \$95 per share.

$$\begin{aligned} T &= 1/4, \quad t = 0, \quad r = 0.08, \quad \sigma = 0.45, \\ \delta &= 0.065, \quad S = 97, \quad K = 95. \end{aligned}$$

Using the formulas for w and P_e^δ we obtain

$$\begin{aligned} w &\approx 0.221763 \\ P_e^\delta &\approx 7.34 \end{aligned}$$

A New Greek and Another Rho

The rate of change in the price of a European call option on a stock paying continuous dividends is

$$\rho_C^\delta = \frac{\partial C_e^\delta}{\partial \delta} = -S(T-t)e^{-\delta(T-t)}\Phi(w).$$

For a European put option

$$\rho_P^\delta = \frac{\partial P_e^\delta}{\partial \delta} = S(T-t)e^{-\delta(T-t)}(1-\Phi(w)).$$

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Remark: some authors call this Greek, Psi, and denote it Ψ.

Dividends Influence on Other Greeks

The presence of the continuous dividend rate δ, in the Call and Put formulas alters the previously discussed Greeks.

$$
\begin{aligned}
w &= \frac{\ln(S/K) + (r - \delta + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}} \\
C_e^\delta &= e^{-\delta(T-t)} S \Phi(w) - K e^{-r(T-t)} \Phi\left(w - \sigma\sqrt{T - t}\right) \\
P_e^\delta &= K e^{-r(T-t)} \Phi\left(w - \sigma\sqrt{T - t}\right) - e^{-\delta(T-t)} S \Phi(-w)
\end{aligned}
$$

Find Delta, Gamma, Rho, Theta, and Vega.

Delta

For a Call:

$$
\begin{aligned}
\frac{\partial C_e^\delta}{\partial S} &= e^{-\delta(T-t)}\Phi\left(w\right) + e^{-\delta(T-t)}\phi\left(w\right)\frac{\partial w}{\partial S} \\
&\quad - Ke^{-r(T-t)}\phi\left(w - \sigma\sqrt{T-t}\right)\frac{\partial w}{\partial S} \\
&= e^{-\delta(T-t)}\Phi\left(w\right).
\end{aligned}
$$

Delta

For a Call:

$$\begin{aligned}\frac{\partial C_e^\delta}{\partial S} &= e^{-\delta(T-t)}\Phi\left(w\right) + e^{-\delta(T-t)}\phi\left(w\right)\frac{\partial w}{\partial S} \\ &\quad - Ke^{-r(T-t)}\phi\left(w - \sigma\sqrt{T-t}\right)\frac{\partial w}{\partial S} \\ &= e^{-\delta(T-t)}\Phi\left(w\right).\end{aligned}$$

For a Put:

$$\begin{aligned}\frac{\partial P_e^\delta}{\partial S} &= Ke^{-r(T-t)}\Phi\left(w - \sigma\sqrt{T-t}\right)\frac{\partial w}{\partial S} - e^{-\delta(T-t)}\Phi\left(-w\right) \\ &\quad + e^{-\delta(T-t)}S\phi\left(-w\right)\frac{\partial w}{\partial S} \\ &= -e^{-\delta(T-t)}\Phi\left(-w\right).\end{aligned}$$

Gamma

For a Call:

$$\begin{aligned}\frac{\partial^2 C_e^\delta}{\partial S^2} &= e^{-\delta(T-t)}\phi(w)\frac{\partial w}{\partial S} \\ &= e^{-\delta(T-t)}\frac{\phi(w)}{\sigma S\sqrt{T-t}}.\end{aligned}$$

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For a Call:

$$\frac{\partial^2 C_e^\delta}{\partial S^2} = e^{-\delta(T-t)}\phi(w)\frac{\partial w}{\partial S} = e^{-\delta(T-t)}\frac{\phi(w)}{\sigma S\sqrt{T-t}}.$$

For a Put:

$$\frac{\partial^2 P_e^\delta}{\partial S^2} = e^{-\delta(T-t)}\phi(-w)\frac{\partial w}{\partial S} = e^{-\delta(T-t)}\frac{\phi(w)}{\sigma S\sqrt{T-t}}.$$

Rho

For a Call:

$$\begin{aligned}\frac{\partial C_e^\delta}{\partial r} &= e^{-\delta(T-t)} S\phi(w)\frac{\partial w}{\partial r} + K(T-t)e^{-r(T-t)}\Phi\left(w - \sigma\sqrt{T-t}\right) \\ &\quad - Ke^{-r(T-t)}\phi\left(w - \sigma\sqrt{T-t}\right)\frac{\partial w}{\partial r} \\ &= K(T-t)e^{-r(T-t)}\Phi\left(w - \sigma\sqrt{T-t}\right).\end{aligned}$$

Rho

For a Call:

$$\begin{aligned}\frac{\partial C_e^\delta}{\partial r} &= e^{-\delta(T-t)} S\phi(w)\frac{\partial w}{\partial r} + K(T-t)e^{-r(T-t)}\Phi\left(w-\sigma\sqrt{T-t}\right) \\ &\quad - Ke^{-r(T-t)}\phi\left(w-\sigma\sqrt{T-t}\right)\frac{\partial w}{\partial r} \\ &= K(T-t)e^{-r(T-t)}\Phi\left(w-\sigma\sqrt{T-t}\right).\end{aligned}$$

For a Put:

$$\begin{aligned}\frac{\partial P_e^\delta}{\partial r} &= -K(T-t)e^{-r(T-t)}\Phi\left(w-\sigma\sqrt{T-t}\right) \\ &\quad + Ke^{-r(T-t)}\phi\left(w-\sigma\sqrt{T-t}\right)\frac{\partial w}{\partial r} + e^{-\delta(T-t)} S\phi(-w)\frac{\partial w}{\partial r} \\ &= -K(T-t)e^{-r(T-t)}\Phi\left(\sigma\sqrt{T-t}-w\right).\end{aligned}$$

For a Call:

$$\begin{aligned}
\frac{\partial C_e^\delta}{\partial t} &= \delta e^{-\delta(T-t)} S\Phi(w) + e^{-\delta(T-t)} S\phi(w) \frac{\partial w}{\partial t} \\
&\quad - Kre^{-r(T-t)}\Phi\left(w - \sigma\sqrt{T-t}\right) \\
&\quad - Ke^{-r(T-t)}\phi\left(w - \sigma\sqrt{T-t}\right)\left(\frac{\partial w}{\partial t} + \frac{\sigma}{2\sqrt{T-t}}\right) \\
&= \delta Se^{-\delta(T-t)}\Phi(w) - Kre^{-r(T-t)}\Phi\left(w - \sigma\sqrt{T-t}\right) \\
&\quad - e^{-\delta(T-t)} \frac{\sigma S\phi(w)}{2\sqrt{T-t}}.
\end{aligned}$$

For a Put:

$$
\begin{aligned}
\frac{\partial P_e^\delta}{\partial t} &= Kre^{-r(T-t)}\Phi\left(w - \sigma\sqrt{T-t}\right) \\
&\quad + Ke^{-r(T-t)}\phi\left(w - \sigma\sqrt{T-t}\right)\left(\frac{\partial w}{\partial t} + \frac{\sigma}{2\sqrt{T-t}}\right) \\
&\quad - \delta e^{-\delta(T-t)}S\Phi(-w) + e^{-\delta(T-t)}S\phi(w)\frac{\partial w}{\partial t} \\
&= Kre^{-r(T-t)}\Phi\left(\sigma\sqrt{T-t} - w\right) - \delta Se^{-\delta(T-t)}\Phi(-w) \\
&\quad - e^{-\delta(T-t)}\frac{\sigma S\phi(w)}{2\sqrt{T-t}}.
\end{aligned}
$$

Vega

For a Call:

$$\begin{aligned}\frac{\partial C_e^\delta}{\partial \sigma} &= e^{-\delta(T-t)} S \phi(w) \frac{\partial w}{\partial \sigma} \\ &\quad - K e^{-r(T-t)} \phi\left(w - \sigma\sqrt{T-t}\right)\left(\frac{\partial w}{\partial \sigma} - \sqrt{T-t}\right) \\ &= S e^{-\delta(T-t)} \sqrt{T-t}\,\phi(w).\end{aligned}$$

Vega

For a Call:

$$\begin{aligned}\frac{\partial C_e^\delta}{\partial \sigma} &= e^{-\delta(T-t)} S\phi(w)\frac{\partial w}{\partial \sigma} \\ &\quad - Ke^{-r(T-t)}\phi\left(w-\sigma\sqrt{T-t}\right)\left(\frac{\partial w}{\partial \sigma}-\sqrt{T-t}\right) \\ &= Se^{-\delta(T-t)}\sqrt{T-t}\,\phi(w).\end{aligned}$$

For a Put:

$$\begin{aligned}\frac{\partial P_e^\delta}{\partial \sigma} &= Ke^{-r(T-t)}\phi\left(w-\sigma\sqrt{T-t}\right)\left(\frac{\partial w}{\partial \sigma}-\sqrt{T-t}\right) \\ &\quad + e^{-\delta(T-t)} S\phi(-w)\frac{\partial w}{\partial \sigma} \\ &= Ke^{-r(T-t)}\sqrt{T-t}\,\phi\left(w-\sigma\sqrt{T-t}\right).\end{aligned}$$

Discrete Dividends

We have argued through absence of arbitrage that a stock must decrease in value by the amount of any dividend paid.

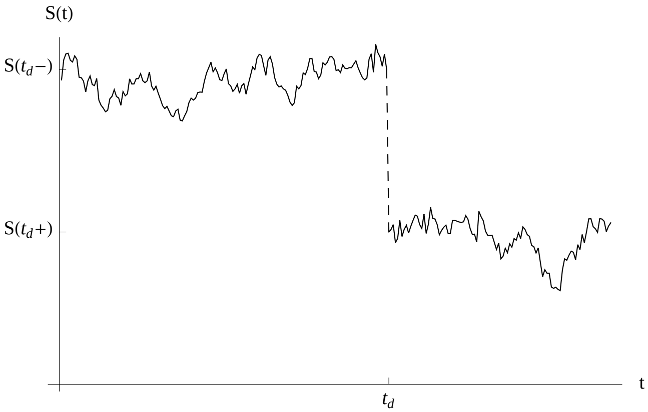
If a stock will pay a proportional dividend $d_y S(t)$ at time t_d then

$$S(t_d^+) = (1 - d_y) S(t_d^-)$$

where

$$\begin{aligned} S(t_d^-) &= \lim_{t \to t_d^-} S(t) \quad \text{and} \\ S(t_d^+) &= \lim_{t \to t_d^+} S(t). \end{aligned}$$

Random Walk



The discontinuous jump in the price of a stock across a dividend date.

Dirac Delta Function

The **Dirac Delta function** $D(t)$, is a function with the following properties:

- $D(t) = 0$ for all $t \neq 0$.
- $\displaystyle\int_{-\infty}^{\infty} D(t)\,dt = 1$
- $\displaystyle\int_{-\infty}^{\infty} f(t)D(t)\,dt = f(0)$ for any continuous function $f(t)$ defined on the real numbers.

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- $\displaystyle\int_{-\infty}^{\infty} f(t) D(t)\, dt = f(0)$ for any continuous function $f(t)$ defined on the real numbers.

Thus if $S(t)$ represents a stock which pays a single discrete dividend at time t_d then S obeys the stochastic differential equation:

$$dS = (\mu - d_y\, D(t - t_d)) S\, dt + \sigma S\, dW(t).$$

Solving for $S(t)$

If $dS = (\mu - d_y\, D\,(t - t_d))S\,dt + \sigma S\,dW(t)$ and $Y = \ln S$ then by Itô's Lemma

$$dY = \left(\mu - d_y D\,(t - t_d) - \frac{1}{2}\sigma^2\right) dt + \sigma\, dW(t)$$

$$Y(t) = Y(0) \begin{cases} \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W(t) & \text{if } t < t_d, \\ \left(\mu - \frac{1}{2}\sigma^2\right) t - d_y + \sigma W(t) & \text{if } t \geq t_d. \end{cases}$$

$$S(t) = S(0) \begin{cases} e^{(\mu - \sigma^2/2)t + \sigma W(t)} & \text{if } t < t_d, \\ e^{(\mu - \sigma^2/2)t - d_y + \sigma W(t)} & \text{if } t \geq t_d. \end{cases}$$

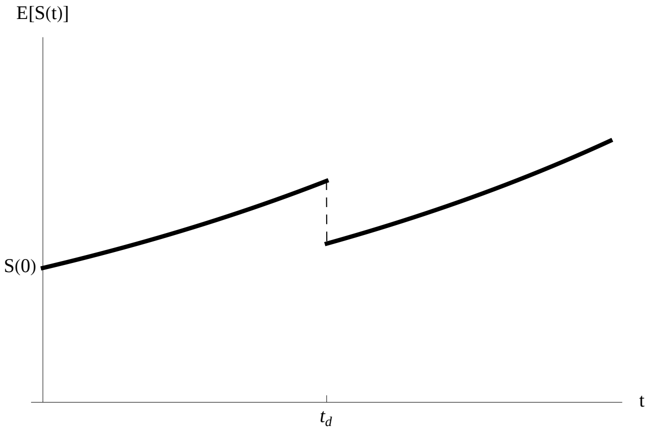
Jump Size

$$
\begin{aligned}
S(t_d^-) - S(t_d^+) &= S(0)e^{(\mu-\sigma^2/2)t_d+\sigma W(t_d)} - S(0)e^{(\mu-\sigma^2/2)t_d-d_y+\sigma W(t_d)} \\
&= S(0)e^{(\mu-\sigma^2/2)t_d+\sigma W(t_d)}\left[1 - e^{-d_y}\right] \\
&= S(t_d^-)\left(1 - e^{-d_y}\right) \\
S(t_d^+) &= S(t_d^-)e^{-d_y}
\end{aligned}
$$

Expected Value Jump

Since $S(t)$ is lognormal,

$$\mathbb{E}\left[S(t)\right] = \begin{cases} S(0)e^{\mu t} & \text{if } t < t_d, \\ S(0)e^{\mu t - d_y} & \text{if } t \geq t_d. \end{cases}$$



Continuity of Option Price (1 of 2)

In the absence of arbitrage, the price of the option must be continuous across the dividend date.

$$\begin{aligned}
\lim_{t \to t_d^-} C^e(S(t), t) &= \lim_{t \to t_d^+} C^e(S(t), t) \\
C^e(S(t_d^-), t_d^-) &= C^e(S(t_d^+), t_d^+) \\
C^e(S(t_d^-), t_d^-) &= C^e(S(t_d^-)e^{-d_y}, t_d^+)
\end{aligned}$$

$$C^e(S(t_d^-), t_d^-) = C^e(S(t_d^-)e^{-d_y}, t_d^+)$$

Remarks:

- The value of the call option will change discontinuously across the dividend date as a function of S.
- However, the price of the option is made continuous by equating the value of the option just before the dividend is paid with the value of the option just after the dividend is paid.
- The value of the stock underlying the option has been adjusted to $S(t_d^+) = S(t_d^-)e^{-d_y}$.

Pricing the Option

Separate the life of the option into two intervals, $[0, t_d)$ and $(t_d, T]$.

- On the interval $(t_d, T]$ no dividends are paid and the original formula for the price of a call option can be used.
- The post dividend value of the stock is Se^{-d_y}.
- For $t_d < t \leq T$,

$$C_e^{\delta} = e^{-d_y} S\Phi(w) - K\Phi\left(w - \sigma\sqrt{T-t}\right)$$

Pre-dividend Pricing

At $t = t_d^+$ the value of the call option is

$$C^e(S(t_d^+), t_d^+) = S(t_d^+)\Phi\left(w\right) - Ke^{-r(T-t_d^+)}\Phi\left(w - \sigma\sqrt{T - t_d^+}\right).$$

Immediately before the dividend date the value of the call option is

$$\begin{aligned} C^e(S(t_d^-), t_d^-) &= C^e(S(t_d^-)e^{-d_y}, t_d^+) \\ &= S(t_d^-)e^{-d_y}\Phi\left(w\right) \\ &\quad - Ke^{-r(T-t_d^+)}\Phi\left(w - \sigma\sqrt{T - t_d^+}\right). \end{aligned}$$

Note: the price of the stock underlying the option has been scaled by the factor e^{-d_y}.

Change of Variable (1 of 2)

Define $\hat{S} = Se^{-d_y}$ then

$$\begin{aligned}\frac{\partial}{\partial S}[F(S,t)] &= e^{-d_y}\frac{\partial}{\partial \hat{S}}[F(S,t)] \quad \text{and} \\ \frac{\partial^2}{\partial S^2}[F(S,t)] &= e^{-2d_y}\frac{\partial^2}{\partial \hat{S}^2}[F(S,t)].\end{aligned}$$

Define $\hat{S} = Se^{-d_y}$ then

$$\begin{aligned}
\frac{\partial}{\partial S}\left[F(S,t)\right] &= e^{-d_y}\frac{\partial}{\partial \hat{S}}\left[F(S,t)\right] \quad \text{and} \\
\frac{\partial^2}{\partial S^2}\left[F(S,t)\right] &= e^{-2d_y}\frac{\partial^2}{\partial \hat{S}^2}\left[F(S,t)\right].
\end{aligned}$$

Substituting these into the Black-Scholes partial differential equation yields

$$rF = F_t + \frac{1}{2}\sigma^2\hat{S}^2F_{\hat{S}\hat{S}} + r\hat{S}F_{\hat{S}}.$$

We have the PDE:

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What about the boundary and final conditions?

$$\begin{aligned}
F(\hat{S}, T) &= (Se^{-d_y} - K)^+ = e^{-d_y}(S - Ke^{d_y})^+ \\
F(0, t) &= F(0 \cdot e^{-d_y}, t) = 0 \\
\lim_{\hat{S} \to \infty} F(\hat{S}, t) &= e^{-d_y}\left(S - Ke^{d_y - r(T-t)}\right)
\end{aligned}$$

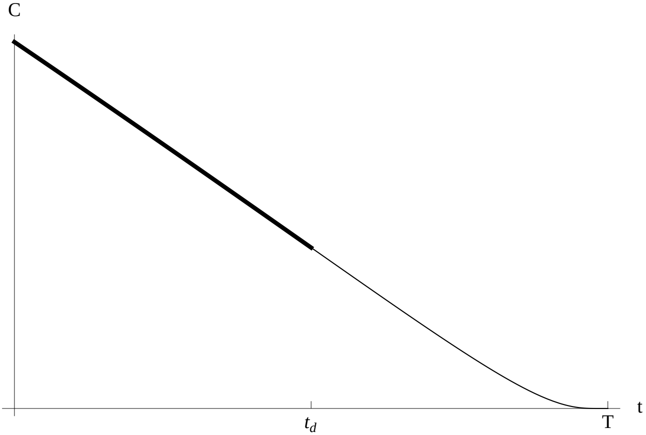
Pre-dividend Pricing

For $0 \leq t < t_d$ use the established European call option pricing formula for e^{-d_y} call options with a strike price of Ke^{d_y}.

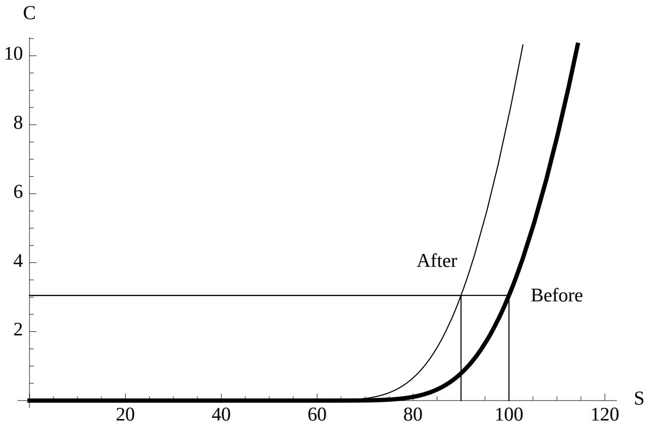
Thus for $t < t_d$,

$$C_e^{\delta} = e^{-d_y}\left[S\Phi\left(w\right) - Ke^{d_y - r(T-t)}\Phi\left(w - \sigma\sqrt{T-t}\right)\right].$$

Continuity of the Option Price (1 of 2)



The value of a European call option on a stock paying a single discrete, proportional dividend at $t = t_d$. The value of the stock is constant where $S(t) = K$, the strike price.



The value of a European call option immediately before (bold) and after a discrete dividend payment. Note that while the value of the stock instantaneously decreases in value by 10% as the dividend is paid, the value of the call option is continuous.

Summary

The value of a European call option written on a stock paying a single discrete, proportional dividend during the life of the option can be written as a piecewise-defined function.

$$C^{e,\delta}(S,t) = \begin{cases} e^{-d_y}\left[S\Phi(w) - Ke^{d_y}\Phi\left(w - \sigma\sqrt{T-t}\right)\right] & \text{if } t < t_d, \\ e^{-d_y}S\Phi(w) - K\Phi\left(w - \sigma\sqrt{T-t}\right) & \text{if } t \geq t_d. \end{cases}$$

Note:

- In the pre-dividend portion of this formula Ke^{d_y} is used for the strike price when calculating w and C_e.
- In the post-dividend formula Se^{-d_y} is used for the value of the underlying stock.

Credits

These slides are adapted from the textbook,

An Undergraduate Introduction to Financial Mathematics, 3rd edition, (2012).

author: J. Robert Buchanan

publisher: World Scientific Publishing Co. Pte. Ltd.

address: 27 Warren St., Suite 401–402, Hackensack, NJ 07601

ISBN: 978-9814407441