

# Forwards and Futures

*An Undergraduate Introduction to Financial Mathematics*

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2014

## Definition

A **forward** is an agreement between two parties to buy or sell a specified quantity of a commodity at a specified price on a specified date in the future.

**Remark:** a forward obligates the parties to buy/sell.

# Example

A restaurant owner needs thirty cases of champagne for a New Year's Eve party. Knowing that a large quantity of champagne may be difficult to obtain (at a reasonable price) at the end of December, the restaurant owner may enter into a forward agreement with a supplier. The terms of the forward would indicate the quantity, price, and terms of delivery for the champagne.

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**Question:** what if the restaurant owner, after entering into the forward agreement cancels the party and thus does not need the champagne?

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**Question:** what if the restaurant owner, after entering into the forward agreement cancels the party and thus does not need the champagne?

**Answer:** the owner may sell the forward contract to someone else in order to recoup his cost.

**Market maker:** an agent who arranges trades between buyers and sellers.

**Bid price:** amount a buyer is willing to spend for an item.

**Ask price:** amount a seller is willing to accept for an item.  
The ask price is also known as the **offer price**.

**Bid/Ask spread:** difference between the bid and ask prices for the same item.

**Dividends:** periodic payments to the owners of a stock paid out of corporate profits.

# Example

- Suppose the lowest ask price of a share of stock is \$50.10 and the highest bid price for the stock is \$50.00.
- The bid/ask spread is therefore \$0.10 per share.
- A stock buyer who issues a buy order for 1000 shares will pay \$50,100.
- The seller will receive \$50,000 and the market maker will earn \$100 on the trade (plus any other fees or commissions charged).

# Process of Buying a Stock

## Three steps:

- 1 Buyer and seller agree on the price for the stock,
- 2 buyer makes payment for the stock,
- 3 seller transfers ownership of the stock to the buyer.



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- 2 buyer makes payment for the stock,
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**Remark:** these three events do not have to occur at the same time.

**outright purchase:** three events occur simultaneously

**fully leveraged purchase:** set price and receive ownership at  $t = 0$ , pay for purchase at  $t = T > 0$ .

**prepaid forward:** set price and pay for purchase at  $t = 0$ , receive ownership at  $t = T > 0$ .

**forward contract:** set price at  $t = 0$ , pay for purchase and receive ownership at  $t = T > 0$ .

# Long and Short Positions

**long position:** the owner of an item is in the “long position”

**short position:** the seller of an item is in the “short position”

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**Remark:** it is common for agents in the market to borrow stocks to sell now with plans to re-purchase later (to return what was borrowed) when the price is lower (they hope).

## Theorem (Outright Purchase)

*If a stock is worth  $S(0)$  at time  $t = 0$  and payment and transfer of ownership will take place at time  $t = 0$  then the amount paid should be  $S(0)$ .*

## Theorem (Outright Purchase)

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## Theorem (Fully Leveraged Purchase)

*If the continuously compounded interest rate is  $r$ , if a stock is worth  $S(0)$  at time  $t = 0$ , transfer of ownership will take place at time  $t = 0$ , and payment will be made at time  $t = T > 0$ , the amount of payment will be  $S(0)e^{rT}$ .*

# Example

- Agent  $A$  believes the price of a stock will decrease during the next 30 days.
- $A$  borrows from  $B$  a share of stock and sells it for  $S(0)$  with the agreement that the stock must be returned to  $B$  by  $t = 30$ .
- If  $S(t) < S(0)$  for some  $0 < t \leq 30$ ,  $A$  purchases the stock for  $S(t)$  and returns it to  $B$ .
- $A$  keeps a net profit of  $S(0) - S(t) > 0$ .

## Theorem

*The price  $F$  of a prepaid forward contract on a non-dividend paying stock initially worth  $S(0)$  at time  $t = 0$  for which ownership of the stock will be transferred to the buyer at time  $t = T > 0$  is  $F = S(0)$ .*



**Assumption:**  $F < S(0)$ .

- 1 Purchase the forward and sell the security. Since  $S(0) - F > 0$ , there is a positive cash flow at  $t = 0$ .
- 2 At  $t = T$ , the buyer receives ownership of the security and immediately closes their short position in the security. The cash flow at  $t = T$  is therefore zero.
- 3 Thus the total cash flows at  $t = 0$  and  $t = T$  is  $S(0) - F > 0$ .

There is no risk since the forward obligates the seller to deliver the security to the buyer so that the buyer's short position in the security can be closed out.

**Assumption:**  $F > S(0)$ .

- 1 Purchase the security at time  $t = 0$  and sell a prepaid forward. Since  $F - S(0) > 0$  there is a positive cash flow at  $t = 0$ .
- 2 At  $t = T$ , the buyer must transfer ownership of the security to the party who purchased the forward. The cash flow at  $t = T$  is therefore zero.
- 3 Thus the total cash flows at  $t = 0$  and  $t = T$  is  $F - S(0) > 0$ .

There is no risk in this situation since the buyer owns the security at time  $t = 0$  and thus will with certainty be able to transfer ownership at  $t = T$ .

# Present Value Proof (1 of 2)

Assume the security's growth rate  $\mu = r$ , the risk-free interest rate.

$$\begin{aligned}dS &= rS dt + \sigma S dW(t) \\dY &= \left(r - \frac{1}{2}\sigma^2\right) dt + \sigma dW(t)\end{aligned}$$

where  $Y = \ln S$ .

$$\begin{aligned}Y(t) - Y(0) &= \left(r - \frac{1}{2}\sigma^2\right) t + \sigma W(t) \\S(t) &= S(0)e^{(r-\sigma^2/2)t + \sigma W(t)}\end{aligned}$$

# Present Value Proof (2 of 2)

Since

$$S(T) = S(0)e^{(r-\sigma^2/2)T+\sigma W(T)}$$

then

$$\mathbb{E}[S(T)] = S(0)e^{(r-\sigma^2/2+\sigma^2/2)T} = S(0)e^{rT}$$

The price of the forward  $F$  should be the present value of the expected value of  $S(T)$ .

$$F = S(0)e^{rT}e^{-rT} = S(0)$$

## Theorem

*Suppose a share of a non-dividend paying stock is worth  $S(0)$  at time  $t = 0$  and that the continuously compounded risk-free interest rate is  $r$ , then the price of the forward contract is*

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**Remark:** dividends are periodic disbursements made to security owners intended to distribute corporate profits to stock holders.

**Assumption:**  $F < S(0)e^{rT}$ .

- 1 The buyer can purchase the forward (which they will not have to pay for until  $t = T$ ) and sell the security at time  $t = 0$ .
- 2 The value of the security is  $S(0)$  which is lent out at the risk-free rate compounded continuously. Thus the net cash flow at time  $t = 0$  is  $S(0) - S(0) = 0$ . At  $t = T$ , when the borrower repays the loan, the buyer's cash balance is  $S(0)e^{rT}$ .
- 3 The buyer pays  $F$  for the forward in order to receive the security which is then used to close out the forward position. The cash flow at  $t = T$  is therefore  $-F$ . Thus the total cash flows at  $t = 0$  and  $t = T$  is  $S(0)e^{rT} - F > 0$ .

There is no risk in obtaining this positive profit since the forward obligates the seller to deliver the security to the buyer so that the buyer's short position in the security can be closed out.

**Assumption:**  $F > S(0)e^{rT}$ .

- 1 The buyer can sell a forward contract which will be paid for at time  $t = T$  and borrow  $S(0)$  to purchase the security at time  $t = 0$ . Thus the net cash flow at time  $t = 0$  is  $S(0) - S(0) = 0$ .
- 2 At  $t = T$ , the buyer must repay the loan of  $S(0)e^{rT}$  and will sell the security for  $F$ . The cash flow at  $t = T$  is therefore  $F - S(0)e^{rT} > 0$ . Thus the total cash flows at  $t = 0$  and  $t = T$  is  $F - S(0)e^{rT} > 0$ .

There is no risk in this situation since the buyer owns the security at time  $t = 0$  and thus will with certainty be able to transfer ownership at  $t = T$ .



## Definition

The **profit** on a forward contract is

$$\text{profit} = S(T) - S(0)e^{rT}.$$

This is the net amount of money gained/lost when the stock is sold on the delivery date.

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**Remark:** we subtract the *future value* of  $S(0)$ , **not** the *present value* of  $S(0)$ . We will apply the same principle later when calculating the profit on any financial position taken by an investor.

# Example

Suppose a share of stock is currently trading for \$25 and the risk-free interest rate is 4.65% per year. Find the price of a two-month forward contract.

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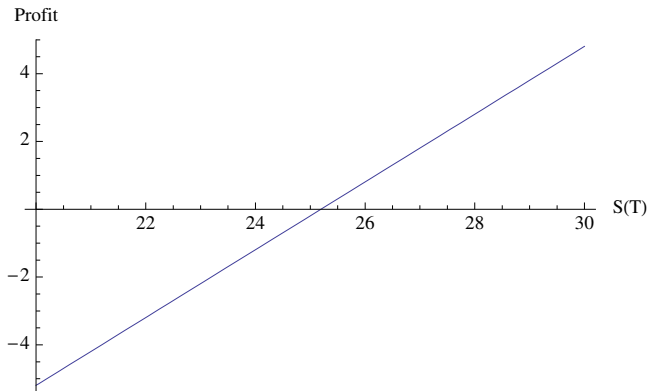
Suppose a share of stock is currently trading for \$25 and the risk-free interest rate is 4.65% per year. Find the price of a two-month forward contract.

The price of a two-month forward contract is

$$F = 25e^{0.0465(2/12)} \approx 25.1945.$$

The profit is then  $S(2/12) - 25.1945$ .

# Illustration



## Remarks:

- Dividends are paid to the *shareholders*, not to the owners of prepaid forwards or forward contracts.
- The price of a prepaid forward or forward contract must be discounted for any dividends paid during the time interval  $[0, T]$ .
- The amount of discount should be the present value of the dividend(s).

# Prepaid Forwards on Dividend Paying Stocks

Assume:

- risk-free interest rate,  $r$
- dividends  $\{D_1, D_2, \dots, D_n\}$  are paid at times  $\{t_1, t_2, \dots, t_n\}$  in the interval  $[0, T]$

Then the price of a prepaid forward on a stock currently valued at  $S(0)$  becomes

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If the stock pays dividends continuously at rate  $\delta$ , then

$$F = S(0)e^{-\delta T}.$$



# Forward Contracts on Dividend Paying Stocks

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If the stock pays dividends continuously at rate  $\delta$ , then

$$F = S(0)e^{(r-\delta)T}.$$

## Example (1 of 4)

Suppose the risk-free interest rate is 5.05%. A share of stock whose current value is \$110 per share will pay a dividend in six months of \$5 and another in twelve months of \$8. Find the prices of a one-year forward contract and one-year prepaid forward on the stock assuming that transfer of ownership will take place immediately after the second dividend is paid.

## Example (2 of 4)

The value of the prepaid forward is

$$F = 110 - 5e^{-0.0505(6/12)} - 8e^{-0.0505(12/12)} \approx 97.5186.$$

The value of a forward contract on the dividend paying stock is

$$F = 97.5186e^{0.0505(12/12)} \approx 102.57.$$

## Example (3 of 4)

An investment valued at \$125 pays dividends continuously at the annual rate of 2.75%. The risk-free interest rate is 3.5%. Find the prices of a four-month prepaid forward and a four-month forward contract on the investment.

## Example (4 of 4)

The price of a four-month prepaid forward on the investment is

$$F = 125e^{-0.0275(4/12)} \approx 123.859.$$

The value of a four-month forward contract on the investment is

$$F = 125e^{(0.035-0.0275)(4/12)} \approx 125.313.$$

# Incorporating Transaction Costs

- $S^a$ : the time  $t = 0$  ask price at which the security can be bought.
- $S^b$ : the time  $t = 0$  bid price at which the security can be sold. In general  $S^b < S^a$ .
- $r^b$ : the continuously compounded interest rate at which money may be borrowed.
- $r^l$ : the continuously compounded interest rate at which money may be lent. In general  $r^l < r^b$ .
- $k$ : the cost per transaction for executing a purchase or sale.

# Pricing a Forward Contract

## Theorem

*The arbitrage-free forward contract price must satisfy the inequality*

$$F^- \equiv (S^b - 2k)e^{r^l T} \leq F \leq (S^a + 2k)e^{r^b T} \equiv F^+.$$



Define  $F^+ = (S^a + 2k)e^{r^b T}$ .

**Assumption:**  $F > F^+$

- 1 At time  $t = 0$  an investor may borrow amount  $S^a + 2k$  to purchase the security and sell the forward contract. The net cash flow at time  $t = 0$  is zero.
- 2 At time  $t = T$  the loan must be repaid in the amount of  $(S^a + 2k)e^{r^b T}$  and the investor receives  $F$  for the forward. The total cash flow for times  $t = 0$  and  $t = T$  is therefore

$$F - (S^a + 2k)e^{r^b T} = F - F^+ > 0.$$

Now define  $F^- = (S^b - 2k)e^{r^l T}$ .

**Assumption:**  $F < F^-$

- 1 At time  $t = 0$  an investor can purchase the forward contract and sell short the security for  $S^b$ . A transaction cost of  $k$  is paid at time  $t = 0$  for the forward contract and another transaction cost of  $k$  is incurred during the short sale. The net proceeds from the sale are  $S^b - 2k$ . This amount is lent out at interest rate  $r^l$  until time  $t = T$ .
- 2 At time  $t = T$  the investor's cash balance is  $(S^b - 2k)e^{r^l T}$ . The investor pays  $F$  for the forward contract and closes out the short position in the security. Thus the total cash flow at times  $t = 0$  and  $t = T$  is

$$(S^b - 2k)e^{r^l T} - F = F^- - F > 0.$$

## Example (1 of 2)

Suppose the asking price for a certain stock is \$55 per share, the bid price is \$54.50 per share, the fee for buying or selling a share or a forward contract is \$1.50 per transaction, the continuously compounded lending rate is 2.5% per year, and the continuously compounded borrowing rate is 5.5% per year. Find the interval of no-arbitrage prices for a three-month forward contract on the stock.

## Example (2 of 2)

$$\begin{aligned}(S^b - 2k)e^{r^l T} &\leq F \leq (S^a + 2k)e^{r^b T} \\ (54.50 - 2(1.50))e^{0.025(3/12)} &\leq F \leq (55 + 2(1.50))e^{0.055(3/12)} \\ 51.7223 &\leq F \leq 58.8030\end{aligned}$$

**Futures** are similar to forward contracts with the following differences:

- Futures are traded in exchanges, while forward contracts can be set up between any two parties.
- Futures are traded in standardized amounts and with standardized maturity dates, whereas forward contracts can be customized to suit the parties involved.
- Futures are usually settled by an exchange of cash between the parties, while a forward contract may involve physical delivery of some commodity (oil, wheat, *etc.*).
- Futures are considered to have less risk of default since the exchange clearinghouse will require deposits from both parties.

**margin:** a deposit on a futures contract held by the clearinghouse to insure against default.

**maintenance margin:** a minimum level of margin (usually a percentage of the value of the futures contract) required by the clearinghouse.

**margin call:** a request for additional margin funds made by the clearinghouse.

## Extended Example (1 of 4)

Suppose a party purchases 500 futures contracts for \$10 each. The contracts mature in 7 days. The continuously compounded interest rate is 10%. The clearing house requires a maintenance margin of 20%. What is the initial margin deposit?

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The initial margin deposit will be

$$(500)(10)(0.20) = \$1000.$$



## Example (2 of 4)

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- The margin has also increased by the change in the futures price multiplied by the number of contracts.

## Example (2 of 4)

Suppose on day 1, the price of a futures contract has increased to \$10.2927. What has changed?

- The margin deposited on day 0 has earned a day's interest.
- The margin has also increased by the change in the futures price multiplied by the number of contracts.

The margin balance is now

$$1000e^{0.10/365} + (10.2927 - 10)500 = \$1146.62.$$

Since  $1146.62 > (500)(10.2927)(0.20) = 1029.27$  (maintenance margin), no margin call is issued.

## Example (3 of 4)

Daily marking-to-market continues until the futures contract matures.

<b>Day</b>	<b>No. of Contracts</b>	<b>Futures Price</b>	<b>Price Change</b>	<b>Margin Balance</b>	<b>Margin Call</b>
0	500	10.00	—	1,000.00	—
1	500	10.29	0.29	1,146.62	0.00

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2	500	11.78	1.49	1,892.57	0.00

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1	500	10.29	0.29	1,146.62	0.00
2	500	11.78	1.49	1,892.57	0.00
3	500	10.62	-1.16	1,308.74	0.00
4	500	11.12	0.50	1,560.04	0.00
5	500	8.58	-2.54	293.70	564.66
6	500	6.45	-2.13	-206.35	851.72
7	500	6.27	-0.18	553.18	—

## Example (4 of 4)

The profit from the futures contract is calculated at maturity by subtracting the future value of the initial margin from the final margin balance.

$$\text{profit} = 553.18 - (1000)e^{0.10(7/365)} = -447.01$$



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