Options

An Undergraduate Introduction to Financial Mathematics

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Definitions and Terminology

Definition

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strike price: agreed upon price for buying or selling.

expiry: deadline by which the option must be exercised (also known as **exercise time**, **strike time**, and **expiry date**).

call option: an option to buy a security (sometimes just called a **call**).

put option: an option to sell a security (a **put** for short).

Some Types of Options

European option: can only be exercised at expiry.

American option: can be exercised at or before expiry.

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American option: can be exercised at or before expiry.

Other types exist such as **Asian**, **Bermudan**, **look-back**, *etc.*

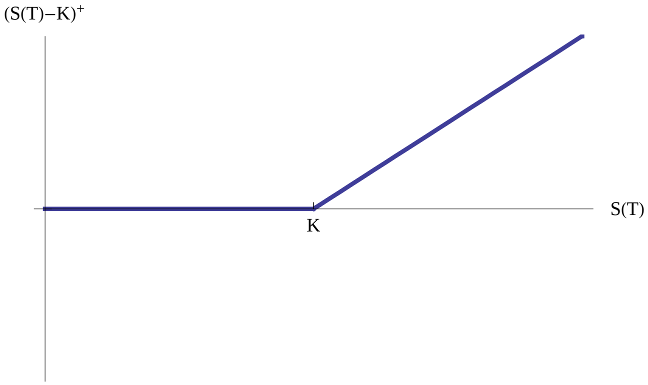
- Our objective is to determine a method for pricing the European-style options.
- Their values satisfy the **Black-Scholes partial differential equation**.

Notation

- C^a: value of an American-style call option
- C^e: value of a European-style call option
- K: strike price of an option
- P^a: value of an American-style put option
- P^e: value of a European-style put option
- r: continuously compounded, risk-free interest rate
- δ: continuously compounded, dividend yield rate
- S: price of a share of a security
- T: exercise time or expiry of an option (sometimes called the strike time)
- t: current time, generally with $0 \leq t \leq T$

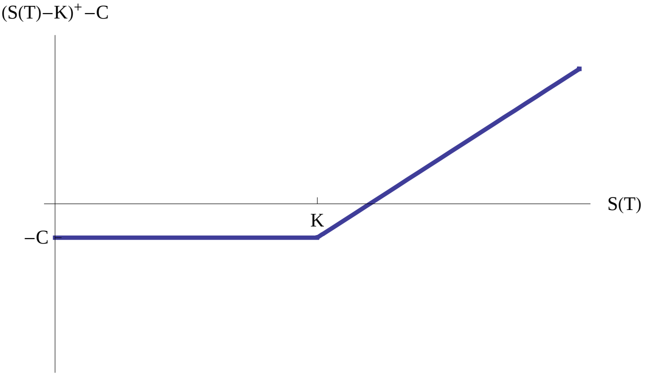
Payoff for a Call Option

A call option does not exhibit a positive payoff until the security price exceeds the strike price.



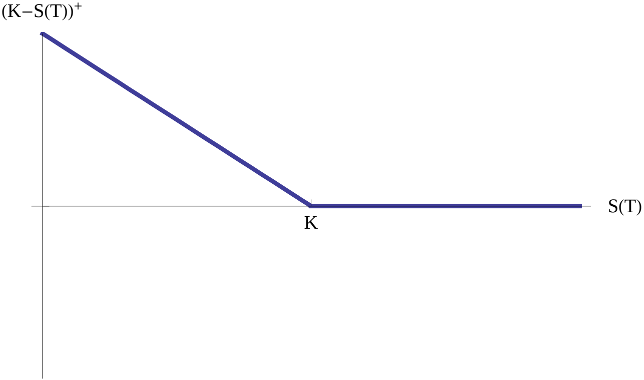
Profit for a Call Option

The payoff of a call option minus its cost is the call's profit.



Payoff for a Put Option

A put option does not exhibit a positive payoff until the strike price exceeds the security price.



Profit for a Put Option

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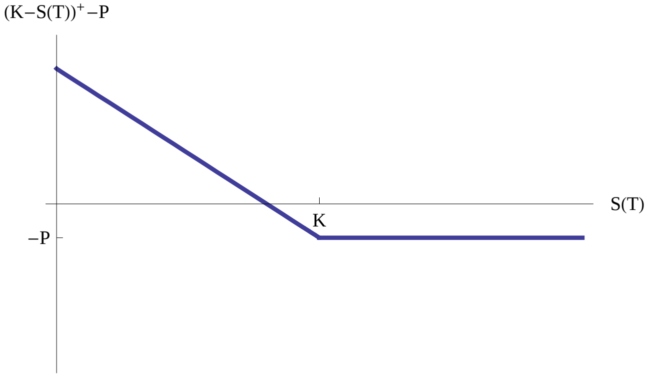

Theorem

$C^a \geq C^e$ and $P^a \geq P^e$.

An American form of an option will always be worth as much as the European version of the option (all other features being the same).

Assume: $C^a < C^e$.

- Sell the European option for C^e and purchase the American option for C^a.
- This generates cash flow $C^e - C^a > 0$ at time $t = 0$ which is invested at the risk-free rate r.
- If the European option holder chooses to exercise the option at expiry, the American option holder can also exercise.
- If the European option holder does not exercise, the American option holder can let the American option expire unused.
- At expiry the seller still holds $(C^e - C^a)e^{rT} > 0$.

Theorem

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Remark: this is equivalent to the inequality

$$C^e + Ke^{-rT} \geq S.$$

It can be interpreted as stating that the cost of a European call plus the present call of the strike is always at least the cost of the security.

Properties of Options (3 of 4)

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Develop a similar inequality relating a put, the strike price, and the security.

Assume: $C^e < S - Ke^{-rT}$.

- Short the security for S and purchase the European call for C^e.
- This generates cash flow $S - C^e$ at time $t = 0$. This is invested at the continuously compounded risk-free rate r.
- At expiry the option holder has a risk-free investment worth $(S - C^e)e^{rT}$ and spends no more than K to close out the short position in the security.
- At expiry the option holder has $(S - C^e)e^{rT} - K > 0$ since this inequality is equivalent to the assumption that $C^e < S - Ke^{-rT}$.
- Therefore a risk-free positive profit can be had.

Put-Call Parity Formula

Theorem

For non-dividend paying stocks, if the European put and call have the same strike price and expiry, then

$$P^e + S = C^e + Ke^{-rT}.$$

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For non-dividend paying stocks, if the European put and call have the same strike price and expiry, then

$$P^e + S = C^e + Ke^{-rT}.$$

Put-Call Parity can be interpreted as stating that the cost of a European put plus the security equals the cost of the European call plus the present value of the strike.

Proof (1 of 2)

Assume: $P^e + S < C^e + Ke^{-rT}$.

- Borrow $P^e + S - C^e$ at the continuously compounded risk-free rate r.
- Purchase the European put option for P^e, the security for S, and sell the European call option for C^e.
- At expiry sell the security for at least K and pay back the loan with interest.
- At expiry this leaves the borrower with $K - (P^e + S - C^e)e^{rT} > 0$ since this inequality is equivalent to the assumption that $P^e + S < C^e + Ke^{-rT}$.
- Therefore a risk-free positive profit can be had.

Proof (2 of 2)

Assume: $P^e + S > C^e + Ke^{-rT}$.

- Short the security for S, sell the European put option for P^e, and purchase the European call option for C^e.
- At time $t = 0$ this generates a cash flow of $S + P^e - C^e > 0$. Invest this amount at the continuously compounded risk-free rate r.
- At expiry purchase the security for at most K and close out the short position.
- At expiry the short seller holds $(S + P^e - C^e)e^{rT} - K > 0$ since this inequality is equivalent to the assumption that $P^e + S > C^e + Ke^{-rT}$.

Effect of Dividends

- Suppose a corporation will pay a dividend to the shareholders at time t_d.
- The amount of the dividend will be $\delta S(t_d)$.

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- Consider the one-sided limits:

$$\begin{aligned} \lim_{t \to t_d^-} S(t) &= S(t_d^-) \\ \lim_{t \to t_d^+} S(t) &= S(t_d^+). \end{aligned}$$

These are the values of the corporation's stock just before and just after the dividend is paid.

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  These are the values of the corporation's stock just before and just after the dividend is paid.
- In the absence of arbitrage,

$$S(t_d^+) = (1 - \delta) S(t_d^-).$$

Theorem

If n dividend payments of the form $\delta S(t_i^-)$ will be made at times t_i^- for $i = 1, 2, \ldots, n$ then the Put-Call Parity Formula for discrete dividend payments can be expressed as

$$P^e + S(0) - \delta \sum_{i=1}^{n} S(t_i^-) e^{-rt_i} = C^e + Ke^{-rT}.$$

Discrete Dividends

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$$P^e + S(0) - \delta \sum_{i=1}^{n} S(t_i^-) e^{-rt_i} = C^e + Ke^{-rT}.$$

The value of the security is discounted by the total of the present values of the dividends paid.

Continuous Dividends

Theorem

For European options on securities which pay dividends at a continuous, constant dividend yield δ, the Put-Call Parity Formula takes on the form

$$P^e + S(0)e^{-\delta T} = C^e + Ke^{-rT}.$$

Binary Model

- Call options can be thought of as insurance against a rise in the price of a security.
- An investor does not know with certainty the value of the security at the strike time, so how much should be paid for the call?

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Example

Suppose $S(0) = \$100$, at $t = T$

$$S(T) = \begin{cases} \$200 & \text{with probability } p, \\ \$50 & \text{with probability } 1 - p. \end{cases}$$

An investor can purchase a European call option whose value is C. The exercise time and strike price of the option are respectively $T = 1$ and \$150. In the absence of arbitrage what is the value of C?

Present Value, Arbitrage-free Setting

- Since two distinct times ($t = 0$ and $t = 1$) are involved we must find the present values of all quantities being compared.
- Assume the simple interest rate for the interval $0 \leq t \leq 1$ is r.
- In the absence of arbitrage, there should be no expected profit from either purchasing the security or the call option.

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- If the expected value of the payoff is 0, then

$$\begin{aligned} 0 &= p\left[-100 + \frac{200}{1+r}\right] + (1-p)\left[-100 + \frac{50}{1+r}\right] \\ p &= \frac{2r+1}{3}. \end{aligned}$$

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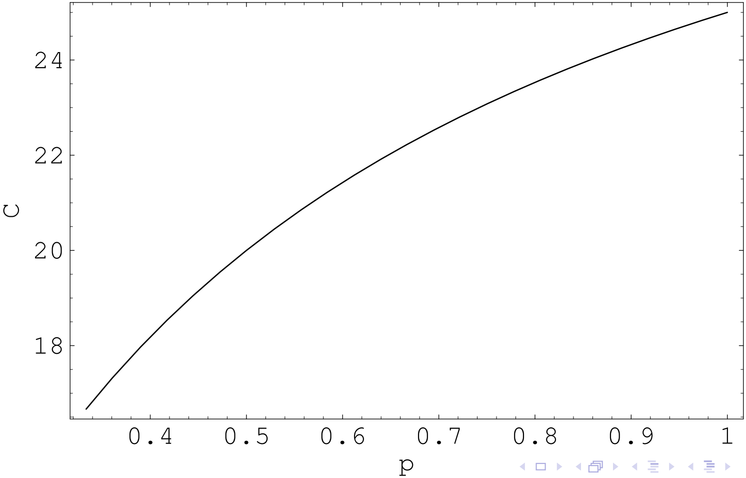
$$\begin{aligned} 0 &= p\left[-C + \frac{(200-150)}{1+r}\right] + (1-p)\left[-C + \frac{(50-150)^+}{1+r}\right] \\ C &= \frac{50(2r+1)}{3(r+1)}. \end{aligned}$$

Parametric Plot of $(p(r), C(r))$

If $(p(r), C(r)) = \left(\frac{2r+1}{3}, \frac{50(2r+1)}{3(r+1)}\right)$ then as the probability of the security reaching \$200 at $T = 1$ increases, so does the cost of the option.



Example (1 of 3)

- Suppose the current value of the stock is $S(0) = \$100$ and at time $T = 1$

$$S(1) = \begin{cases} \$150 & \text{with probability } p = 0.45, \\ \$75 & \text{with probability } 1 - p = 0.55. \end{cases}$$

- Suppose further that the risk-free interest rate is $r = 8.388\%$.
- A European call option with a strike price of \$125 can be purchased for \$10 (note the arbitrage-free price is \$10.3448).

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- Suppose further that the risk-free interest rate is $r = 8.388\%$.
- A European call option with a strike price of \$125 can be purchased for \$10 (note the arbitrage-free price is \$10.3448).
- Design an investment scheme which guarantees a positive profit.

Solution: Borrow funds to take a position of x shares of the stock and y call options (x and y can be positive or negative).

- At $t = 0$, the portfolio is worth $100x + 10y$.
- At $t = T$, the investor owes $(100x + 10y)e^{0.08338}$.
- If $S(1) = 150$, the portfolio generates a cash flow of

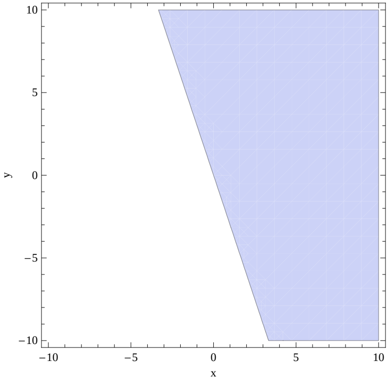
$$150x + (150 - 125)y = 150x + 25y.$$

- If $S(1) = 75$, the portfolio generates a cash flow of $75x$.

A positive profit is guaranteed in the region where

$$\begin{aligned} 150x + 25y &> (100x + 10y)e^{0.08338} \\ 75x &> (100x + 10y)e^{0.08338} \end{aligned}$$

which implies $3x + y > 0$.



We now turn our attention to mathematically modeling the value of a European call option.

- Suppose a stock obeys an Itô process of the form:

$$dS = \mu S\,dt + \sigma S\,dW(t)$$

- An investor will create a portfolio Y, consisting of a short position in a European call option and a long position of Δ shares of the stock.

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- Use Itô's lemma to find the stochastic process followed by Y.

Itô Process for dY

$$
\begin{aligned}
dY &= \left(\mu S Y_S + \frac{1}{2}\sigma^2 S^2 Y_{SS} + Y_t\right) dt + (\sigma S Y_S)\, dW(t) \\
&= \left(\mu S \left[\frac{\partial C^e}{\partial S} - \Delta\right] + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C^e}{\partial S^2} + \frac{\partial C^e}{\partial t}\right) dt \\
&\quad + \sigma S \left(\frac{\partial C^e}{\partial S} - \Delta\right) dW(t)
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Note: the process becomes deterministic if $\Delta = \frac{\partial C^e}{\partial S}$.

$$dY = \left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C^e}{\partial S^2} + \frac{\partial C^e}{\partial t}\right) dt$$

No Arbitrage Assumption

The payoff from the portfolio should be the same as that generated by investing an equivalent amount of money to Y in savings earning interest compounded continuously at rate r.

$$\begin{aligned} dY &= rY\,dt \\ &= r(C^e - (\Delta)S)\,dt \\ &= r\left(C^e - S\frac{\partial C^e}{\partial S}\right)\,dt \end{aligned}$$

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$$dY = \left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C^e}{\partial S^2} + \frac{\partial C^e}{\partial t}\right)\,dt.$$

Equating the two expressions for dY yields the **Black-Scholes partial differential equation**

$$r\,C^e = \frac{\partial C^e}{\partial t} + r\,S\frac{\partial C^e}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C^e}{\partial S^2}.$$

Final and Boundary Conditions (1 of 2)

In order to solve the Black-Scholes PDE we must have some boundary and final conditions.

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At expiry the call option is worth $(S(T) - K)^+$, so

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At expiry the call option is worth $(S(T) - K)^+$, so

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this is the final condition.

The stock will have a value in the interval $[0, \infty)$. The boundary at $S = 0$ is absorbing, so if there is a time $t^* \geq 0$ such that $S(t^*) = 0$, then $S(t) = 0$ for all $t \geq t^*$. In this case the option will never be exercised and is worthless. Thus

$$C^e(0, t) = 0,$$

which is the boundary condition at $S = 0$.

From the Put-Call Parity Formula:

$$\begin{aligned} C^e &= P^e + S - Ke^{-rT} \\ \lim_{S\to\infty} C^e &= \lim_{S\to\infty} P^e + S - Ke^{-rT} \\ C^e &\to S - Ke^{-rT} \quad \text{as } S \to \infty. \end{aligned}$$

Boundary Conditions (2 of 2)

From the Put-Call Parity Formula:

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As the security unbounded in value:

- a put option (right to sell at a finite price) becomes worthless, and
- the call option is worth the difference between the security price and the present value of the strike.

Initial Boundary Value Problem

For (S, t) in $[0, \infty) \times [0, T]$,

$$
\begin{aligned}
rC^e &= \frac{\partial C^e}{\partial t} + rS\frac{\partial C^e}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C^e}{\partial S^2} \\
C^e(S, T) &= (S(T) - K)^+ \quad \text{for } S > 0, \\
C^e(0, t) &= 0 \quad \text{for } 0 \leq t < T, \\
C^e(S, t) &= S - Ke^{-r(T-t)} \quad \text{as } S \to \infty.
\end{aligned}
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The solution of this IBVP is the subject of the next chapter.

Spreads

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A trading strategy involving two or more options of the same type on the same stock is called a **spread**.

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Definition

A long call position with a strike price of K_1 and a short call position with a strike price of $K_2 > K_1$ is called a **bull spread**.

Remark: We are assuming the underlying stock is the same and the expiry of the two calls is the same.

Bull Spread (1 of 2)

- Since $K_2 > K_1$ then $C_2 < C_1$ (why?).
- Initial outlay of capital in amount $C_1 - C_2 > 0$.
- Payoff of long call $(S(T) - K_1)^+$.
- Payoff of short call $-(S(T) - K_2)^+$.

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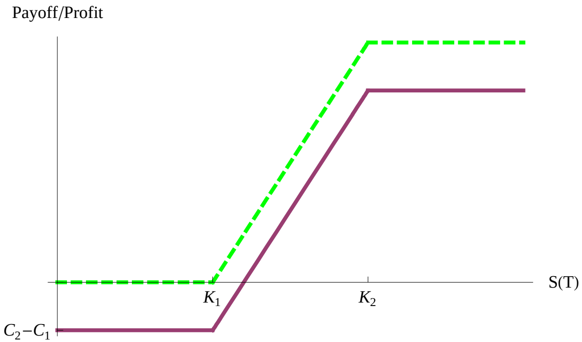
$S(T)$	**Long Call Payoff**	**Short Call Payoff**	**Total Payoff**
$S(T) \leq K_1$	0	0	0
$K_1 < S(T) < K_2$	$S(T) - K_1$	0	$S(T) - K_1$
$K_2 \leq S(T)$	$S(T) - K_1$	$K_2 - S(T)$	$K_2 - K_1$

Bull Spread (2 of 2)



Example: Bull Spread

Example

Suppose we create a bull spread purchasing a call option with strike price \$115 and selling a call option with strike price \$130. Suppose that $C(115) = 5$ and $C(130) = 3$. Find the payoff and net profit if at expiry,

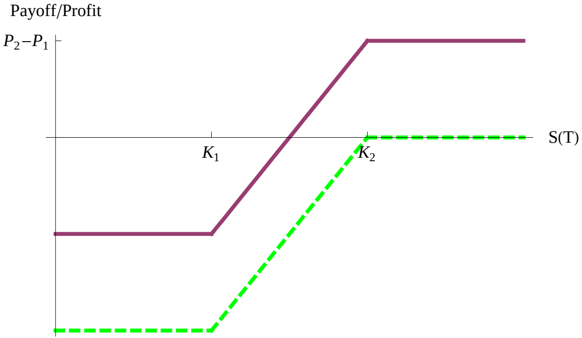
- $S = 110$
- $S = 115$
- $S = 125$
- $S = 130$
- $S = 135$.

Bull Spread with Puts (1 of 2)

Suppose an investor has a long position in a put with strike price K_1 and a short position in a put with strike price $K_2 > K_1$.

$S(T)$	**Long Put Payoff**	**Short Put Payoff**	**Total Payoff**
$S(T) \leq K_1$	$K_1 - S(T)$	$S(T) - K_2$	$K_1 - K_2$
$K_1 < S(T) < K_2$	0	$S(T) - K_2$	$S(T) - K_2$
$K_2 \leq S(T)$	0	0	0

Bull Spread with Puts (2 of 2)



Example

Suppose we create a bull spread purchasing a put option with strike price \$95 and selling a put option with strike price \$105. Suppose that $P(95) = 5$ and $P(105) = 8$. Find the payoff and net profit if at expiry,

- $S = 90$
- $S = 95$
- $S = 100$
- $S = 105$
- $S = 110$.

Bear Spreads

Definition

A short call position with a strike price of K_1 and a long call position with a strike price of $K_2 > K_1$ is called a **bear spread.**

Remarks:

- We are assuming the underlying stock is the same and the expiry of the two calls is the same.
- The positions in the bear spread are opposite those of the bull spread.

Bear Spread (1 of 2)

- Since $K_2 > K_1$ then $C_2 < C_1$.
- Initial income of capital in amount $C_1 - C_2 > 0$.
- Payoff of long call $(S(T) - K_2)^+$.
- Payoff of short call $-(S(T) - K_1)^+$.

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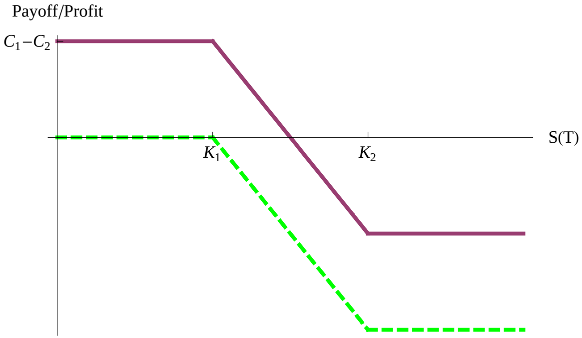
- Payoff of short call $-(S(T) - K_1)^+$.

$S(T)$	**Long Call Payoff**	**Short Call Payoff**	**Total Payoff**
$S(T) \leq K_1$	0	0	0
$K_1 < S(T) < K_2$	0	$K_1 - S(T)$	$K_1 - S(T)$
$K_2 \leq S(T)$	$S(T) - K_2$	$K_1 - S(T)$	$K_1 - K_2$

Bear Spread (2 of 2)



Example: Bear Spread

Example

Suppose we create a bear spread purchasing a call option with strike price \$150 and selling a call option with strike price \$125. Suppose that $C(150) = 5$ and $C(125) = 10$. Find the payoff and net profit if at expiry,

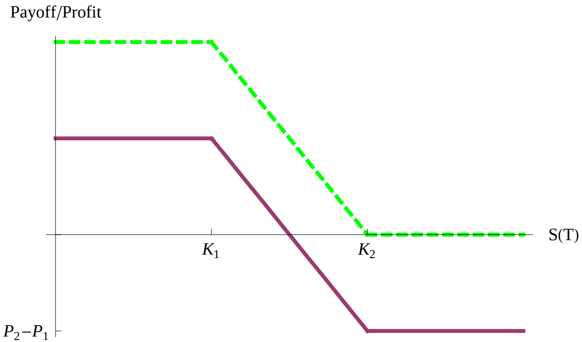
- $S = 120$
- $S = 125$
- $S = 150$
- $S = 150$
- $S = 160$.

Bear Spread with Puts (1 of 2)

Suppose an investor has a short position in a put with strike price K_1 and a long position in a put with strike price $K_2 > K_1$.

$S(T)$	**Long Put Payoff**	**Short Put Payoff**	**Total Payoff**
$S(T) \leq K_1$	$K_2 - S(T)$	$S(T) - K_1$	$K_2 - K_1$
$K_1 < S(T) < K_2$	$K_2 - S(T)$	0	$K_2 - S(T)$
$K_2 \leq S(T)$	0	0	0

Bear Spread with Puts (2 of 2)



Example: Bear Spread

Example

Suppose we create a bear spread selling a put option with strike price \$50 and purchasing a put option with strike price \$60. Suppose that $P(50) = 3$ and $P(60) = 5$. Find the payoff and net profit if at expiry,

- $S = 45$
- $S = 50$
- $S = 55$
- $S = 60$
- $S = 65$.

Definition

A long call position with a strike price of K_1, a long call position with a strike price of $K_3 > K_1$, and a short position in two calls with strike price $K_2 = (K_1 + K_3)/2$ is called a **butterfly spread**.

Remark: We are assuming the underlying stock is the same and the expiry of the three calls is the same.

Butterfly Spread

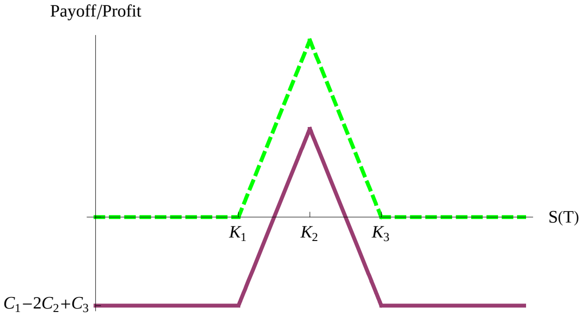
- Note that $K_3 > K_2 > K_1$.
- Payoffs of long calls are $(S(T) - K_1)^+$ and $(S(T) - K_3)^+$.
- Payoff of two short calls $-2(S(T) - K_2)^+$.

Butterfly Spread

- Note that $K_3 > K_2 > K_1$.
- Payoffs of long calls are $(S(T) - K_1)^+$ and $(S(T) - K_3)^+$.
- Payoff of two short calls $-2(S(T) - K_2)^+$.

$S(T)$	**Payoff 1st Long Call**	**Payoff 2nd Long Call**	**Payoff Short Calls**	**Payoff Total**
$S(T) \leq K_1$	0	0	0	0
$K_1 < S(T) < K_2$	$S(T) - K_1$	0	0	$S(T) - K_1$
$K_2 \leq S(T) < K_3$	$S(T) - K_1$	0	$-2(S(T) - K_2)$	$K_3 - S(T)$
$K_3 \leq S(T)$	$S(T) - K_1$	$S(T) - K_3$	$-2(S(T) - K_2)$	0

Butterfly Spread with Calls



Example: Butterfly Spread

Example

Suppose we create a butterfly spread purchasing a call options with strike prices \$150 and \$200 selling 2 call options with strike price \$175. Suppose that $C(150) = 60$, $C(175) = 35$, and $C(200) = 10$. Find the payoff and net profit if at expiry,

- $S = 100$
- $S = 150$
- $S = 175$
- $S = 200$
- $S = 250.$

Definition

Simultaneous long position in a call and a put is called a **long straddle**. Simultaneous short position in a call and a put is called a **short straddle**.

Remarks: We will assume

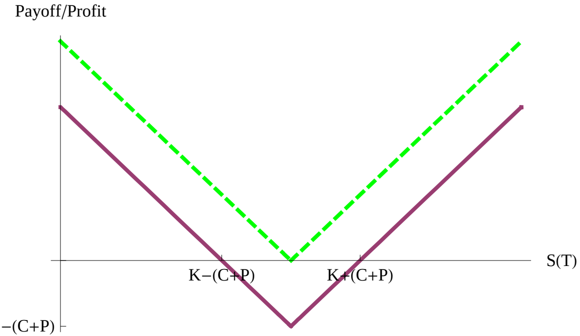
- the underlying stock is the same for both options,
- the strike prices are the same,
- the expiry dates are the same.

Long Straddle (1 of 2)

Suppose an investor has a long straddle with strike prices K.

$S(T)$	**Put Payoff**	**Call Payoff**	**Total Payoff**
$S(T) \leq K$	$K - S(T)$	0	$K - S(T)$
$K < S(T)$	0	$S(T) - K$	$S(T) - K$

Long Straddle (2 of 2)



Example: Long Straddle

Example

Suppose we create a long straddle purchasing call and put options with strike prices of \$150. Suppose that $C(150) = 20$ and $P(150) = 10$. Find the payoff and net profit if at expiry,

- $S = 100$
- $S = 140$
- $S = 150$
- $S = 160$
- $S = 200$.

Strangles

Definition

A **long strangle** is a simultaneous long position in a call and a put. A **short strangle** is a simultaneous short position in a call and a put.

Remarks: We will assume

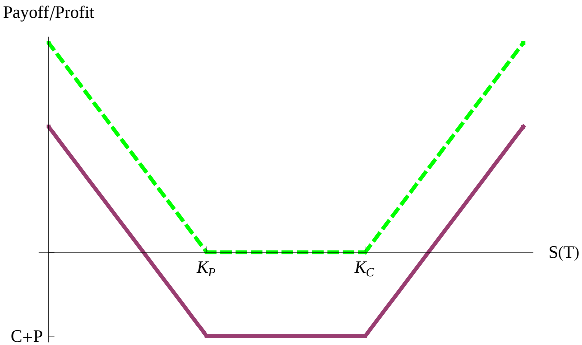
- the underlying stock is the same for both options,
- the expiry dates are the same,
- the strike prices may be different (this characteristic distinguishes a strangle from a straddle).

Long Strangle (1 of 2)

Suppose an investor has a long strangle with strike prices K_P for the put and $K_C > K_P$ for the call.

$S(T)$	**Put Payoff**	**Call Payoff**	**Total Payoff**
$S(T) \leq K_P$	$K_P - S(T)$	0	$K_P - S(T)$
$K_P < S(T) < K_C$	0	0	0
$K_C < S(T)$	0	$S(T) - K_C$	$S(T) - K_C$

Long Strangle (2 of 2)



Example: Long Strangle

Example

Suppose we create a long strangle purchasing a call option with a strike price of \$150 and a put option with a strike price of \$130. Suppose that $C(150) = 20$ and $P(130) = 10$. Find the payoff and net profit if at expiry,

- $S = 100$
- $S = 140$
- $S = 150$
- $S = 160$
- $S = 200$.

Credits

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