Forcing of Solutions to Reaction-Diffusion Equations With Applications to Population Models

J. Robert Buchanan

Bob.Buchanan@millersville.edu

Millersville University of Pennsylvania

Introduction

Initial-boundary value problem:

$$(1)\qquad \mathbf{u}_t = \mathbf{f}(\mathbf{u}) + \mathbf{g}(\mathbf{x},t) + D\Delta\mathbf{u} \qquad \text{on } \Omega \times (0,\infty)$$

$$(2)\qquad \frac{\partial \mathbf{u}}{\partial n}(\mathbf{x},t) = 0 \qquad \text{on } \partial\Omega \times (0,\infty)$$

$$(3)\qquad \mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}) \qquad \text{on } \Omega$$

- Reaction-diffusion equation's solution converges to a spatially homogeneous solution
- Analysis of the asymptotic behavior simplified

Goal

- Find solution to a system of ODEs with spatially averaged reaction terms.
- Show this solution is stable as a solution to the reaction-diffusion equations.
- Show that when the diffusion coefficients are sufficiently large, solutions are asymptotically homogeneous in Ω.

Spatially Averaged Equation

Solution to (1)–(3) will be compared to the solution of:

(4) $$\mathbf{v}_t = \mathbf{f}(\mathbf{v}) + \overline{\mathbf{g}}(t) \qquad \text{for } t > 0$$

(5) $$\mathbf{v}(0) = \frac{1}{|\Omega|}\int_\Omega \mathbf{u}_0(\mathbf{x})\, d\Omega$$

where

$$\overline{\mathbf{g}}(t) = \frac{1}{|\Omega|}\int_\Omega \mathbf{g}(\mathbf{x}, t)\, d\Omega$$

and $|\Omega|$ is the measure of the spatial domain Ω.

Stability

Definition 1 *A bounded solution* $\mathbf{v}(t)$*, to (4) and (5) will be said to be stable as a solution to (1) and (2) if for every* $\epsilon > 0$ *there exists a* $\delta > 0$ *and a time* $T > 0$ *such that*

$$\|\mathbf{u}(\mathbf{x},0) - \mathbf{v}(0)\|_{L^2(\Omega)} < \delta$$

implies

$$\|\mathbf{u}(\mathbf{x},t) - \mathbf{v}(t)\|_{L^2(\Omega)} \leq \epsilon$$

for all $t \geq T$*, where*

$$\|\cdot\|^2_{L^2(\Omega)} = \int_\Omega \langle \cdot, \cdot \rangle \, d\Omega$$

Stability Result

Theorem 2 *Suppose there exists a compact positively invariant region* $\Sigma \subset \mathbb{R}^n$ *for (1) with* $\mathbf{u}_0(\mathbf{x}) \in \Sigma$ *and suppose* $\mathbf{v}(t)$ *is a solution to (4) such that there exists a positive constant* γ *for which* $\langle \mathbf{f_u}(\mathbf{v}(t))\mathbf{w}, \mathbf{w} \rangle \leq -\gamma |\mathbf{w}|^2$ *for all* $\mathbf{w} \in \Sigma$*, then if* d*, the smallest diffusion coefficient of* D *is sufficiently large,* $\mathbf{v}(t)$ *is stable as a solution to (1) and (2).*

Outline of Proof

1. Bound the deviation of the solution to (1)–(3) from its spatial average over domain Ω.
2. Bound the deviation of the spatial average from the solution to (4) and (5).
3. Use the Minkowski inequality to combine those two bounds into a bound on the deviation of the solution to (1)–(3) from that of (4) and (5).

Spatially Averaged Solution

Spatial average of solution to (1)–(3) is

$$\overline{\mathbf{u}}(t) = |\Omega|^{-1} \int_\Omega \mathbf{u}(\mathbf{x}, t)\, d\Omega,$$

Solves the following initial value problem for $t > 0$.

$$\overline{\mathbf{u}}_t = \mathbf{f}(\overline{\mathbf{u}}) + \overline{\mathbf{g}}(t) + \frac{1}{|\Omega|} \int_\Omega \mathbf{f}(\mathbf{u}) - \mathbf{f}(\overline{\mathbf{u}})\, d\Omega \tag{6}$$

$$\overline{\mathbf{u}}(0) = \frac{1}{|\Omega|} \int_\Omega \mathbf{u}_0(\mathbf{x})\, d\Omega \tag{7}$$

Energy Method

Let $\phi(t) = (1/2)\|\mathbf{u}(\mathbf{x},t) - \overline{\mathbf{u}}(t)\|^2_{L^2(\Omega)}$, then

$$\begin{aligned} \frac{d\phi}{dt} &= \int_\Omega \langle \mathbf{u}_t - \overline{\mathbf{u}}_t, \mathbf{u} - \overline{\mathbf{u}} \rangle \, d\Omega \\ &= \int_\Omega \langle \mathbf{f}(\mathbf{u}) - \mathbf{f}(\overline{\mathbf{u}}), \mathbf{u} - \overline{\mathbf{u}} \rangle \, d\Omega + \\ &\quad \int_\Omega \langle \mathbf{g}(\mathbf{x},t) - \overline{\mathbf{g}}(t), \mathbf{u} - \overline{\mathbf{u}} \rangle \, d\Omega + \\ &\quad \int_\Omega \left\langle \frac{1}{|\Omega|} \int_\Omega \mathbf{f}(\mathbf{u}) - \mathbf{f}(\overline{\mathbf{u}}) \, d\Omega, \mathbf{u} - \overline{\mathbf{u}} \right\rangle \, d\Omega + \\ &\quad \int_\Omega \langle D\Delta\mathbf{u}, \mathbf{u} - \overline{\mathbf{u}} \rangle \, d\Omega \end{aligned}$$

Diffusion Integral

$$\int_\Omega \langle D\Delta\mathbf{u}, \mathbf{u}-\overline{\mathbf{u}}\rangle\, d\Omega = -\int_\Omega \langle D\nabla(\mathbf{u}-\overline{\mathbf{u}}), \nabla(\mathbf{u}-\overline{\mathbf{u}})\rangle\, d\Omega$$
$$\leq -d\|\nabla(\mathbf{u}-\overline{\mathbf{u}})\|^2_{L^2(\Omega)}$$

Since $\overline{\mathbf{u}}$ is independent of $\mathbf{x}$, by Green's first identity, and the boundary conditions of equation (2). Then

(8) $$\int_\Omega \langle D\Delta\mathbf{u}, \mathbf{u}-\overline{\mathbf{u}}\rangle\, d\Omega \leq -d\lambda\|\mathbf{u}-\overline{\mathbf{u}}\|^2_{L^2(\Omega)} = -2d\lambda\phi(t),$$

λ is the smallest positive eigenvalue of $-\Delta$ with homogeneous Neumann boundary conditions on Ω.

Reaction Term Integral

$$\begin{aligned}\langle \mathbf{f}(\mathbf{u}) - \mathbf{f}(\overline{\mathbf{u}}), \mathbf{u} - \overline{\mathbf{u}} \rangle &\leq \frac{1}{2}|\mathbf{f}(\mathbf{u}) - \mathbf{f}(\overline{\mathbf{u}})|^2 + \frac{1}{2}|\mathbf{u} - \overline{\mathbf{u}}|^2 \\ &\leq \frac{M^2}{2}|\mathbf{u} - \overline{\mathbf{u}}|^2 + \frac{1}{2}|\mathbf{u} - \overline{\mathbf{u}}|^2,\end{aligned}$$

where $M = \max_{\mathbf{u} \in \Sigma}\{|\mathbf{f}_{\mathbf{u}}|\}$. Thus the integral below involving the reaction term of (1) obeys the inequality:

(9) $$\int_\Omega \langle \mathbf{f}(\mathbf{u}) - \mathbf{f}(\overline{\mathbf{u}}), \mathbf{u} - \overline{\mathbf{u}} \rangle \, d\Omega \leq (M^2 + 1)\phi(t).$$

Forcing Term Integral

$$\langle \mathbf{g}(\mathbf{x},t)-\overline{\mathbf{g}}(t),\mathbf{u}-\overline{\mathbf{u}}\rangle \leq \frac{1}{2}|\mathbf{g}(\mathbf{x},t)-\overline{\mathbf{g}}(t)|^2+\frac{1}{2}|\mathbf{u}-\overline{\mathbf{u}}|^2$$

Components of $\mathbf{g}$ and $\overline{\mathbf{g}}$ are periodic in t, and $\mathbf{g}$ is continuous in $\mathbf{x}$, thus there exists $K > 0$ such that

$$\|\mathbf{g}(\mathbf{x},t)-\overline{\mathbf{g}}(t)\|^2_{L^2(\Omega)} \leq K^2.$$

Therefore

(10) $$\int_\Omega \langle \mathbf{g}(\mathbf{x},t)-\overline{\mathbf{g}}(t),\mathbf{u}-\overline{\mathbf{u}}\rangle \, d\Omega \leq \frac{K^2}{2}+\phi(t)$$

Averaged Integral

$$\int_\Omega \left\langle \int_\Omega \mathbf{f}(\mathbf{u}) - \mathbf{f}(\overline{\mathbf{u}})\, d\Omega, \mathbf{u} - \overline{\mathbf{u}} \right\rangle d\Omega =$$

(11) $$\left\langle \int_\Omega \mathbf{f}(\mathbf{u}) - \mathbf{f}(\overline{\mathbf{u}})\, d\Omega, \int_\Omega \mathbf{u} - \overline{\mathbf{u}}\, d\Omega \right\rangle = 0$$

Finally we have

(12) $$\frac{d\phi}{dt} + (2d\lambda - M^2 - 2)\phi(t) \leq \frac{K^2}{2}$$

Let $\sigma = 2d\lambda - M^2 - 2$.

Differential Inequality

Integrating (12) with respect to time over the interval $[0, t]$ produces

$$\phi(t) \leq \frac{1}{2}\|\mathbf{u}_0(\mathbf{x}) - \overline{\mathbf{u}}(0)\|^2_{L^2(\Omega)} e^{-\sigma t} + \frac{K^2}{2\sigma}(1 - e^{-\sigma t})$$

or equivalently

(13) $$\|\mathbf{u}(\mathbf{x}, t) - \overline{\mathbf{u}}(t)\|^2_{L^2(\Omega)} \leq \|\mathbf{u}_0(\mathbf{x}) - \overline{\mathbf{u}}(0)\|^2_{L^2(\Omega)} e^{-\sigma t} + \frac{K^2}{\sigma}.$$

2nd Step

Define $\psi(t) = (1/2)|\overline{\mathbf{u}}(t) - \mathbf{v}(t)|^2$.

$$\begin{aligned} \frac{d\psi}{dt} &= \langle \overline{\mathbf{u}}_t - \mathbf{v}_t, \overline{\mathbf{u}} - \mathbf{v} \rangle \\ &\leq -\gamma|\overline{\mathbf{u}} - \mathbf{v}|^2 + \frac{1}{|\Omega|}\left\langle \int_\Omega \mathbf{f}(\mathbf{u}) - \mathbf{f}(\overline{\mathbf{u}})\, d\Omega, \overline{\mathbf{u}} - \mathbf{v} \right\rangle \\ &\qquad + \langle \mathbf{f}(\overline{\mathbf{u}}) - \mathbf{f}(\mathbf{v}) - \mathbf{f}_{\mathbf{u}}(\mathbf{v})(\overline{\mathbf{u}} - \mathbf{v}), \overline{\mathbf{u}} - \mathbf{v} \rangle \end{aligned}$$

For every $\delta > 0$ there exists an $\eta \equiv \eta(\delta) > 0$ such that if $|\overline{\mathbf{u}} - \mathbf{v}|^2 < 2\eta$ then

$$|\langle \mathbf{f}(\overline{\mathbf{u}}) - \mathbf{f}(\mathbf{v}) - \mathbf{f}_{\mathbf{u}}(\mathbf{v})(\overline{\mathbf{u}} - \mathbf{v}), \overline{\mathbf{u}} - \mathbf{v} \rangle| < \frac{\delta}{2}|\overline{\mathbf{u}} - \mathbf{v}|^2.$$

Note $\psi(0) = 0$ and $\psi(t)$ is continuous. There exists $t_0 > 0$ for which $\psi(t) < \eta$ on the interval $[0, t_0)$. Thus

$$\frac{d\psi}{dt} \leq \left(\delta - 2\gamma + \frac{\delta}{|\Omega|}\right)\psi(t) + \frac{1}{2\delta|\Omega|}\left(\int_\Omega |\mathbf{f}(\mathbf{u}) - \mathbf{f}(\overline{\mathbf{u}})|\, d\Omega\right)^2.$$

Choose $\delta = \gamma|\Omega|/(|\Omega| + 1)$, use Jensen's inequality, and the inequality in (13), then on the interval $[0, t_0)$,

$$\frac{d\psi}{dt} + \gamma\psi(t) \leq$$

$$\frac{M^2(|\Omega| + 1)}{2\gamma|\Omega|}\left(\|\mathbf{u}_0(\mathbf{x}) - \overline{\mathbf{u}}(0)\|^2_{L^2(\Omega)} e^{-\sigma t} + \frac{K^2}{\sigma}\right) \tag{14}$$

Let $N^2 = (M^2/\gamma)(1 + |\Omega|^{-1})$. Then for $0 \leq t < t_0$,

$$\text{(15)} \quad \psi(t) \leq \frac{N^2}{2(\gamma - \sigma)} \|\mathbf{u}_0(\mathbf{x}) - \overline{\mathbf{u}}(0)\|^2_{L^2(\Omega)} (e^{-\sigma t} - e^{-\gamma t}) + \frac{K^2 N^2}{2\gamma\sigma}(1 - e^{-\gamma t}).$$

Choose $d > (\gamma + M^2 + 2)/(2\lambda)$ then $\sigma > \gamma > 0$ and $0 < e^{-(\sigma - \gamma)t} \leq 1$ for all $t \geq 0$. Thus for $t \in [0, t_0)$

$$\text{(16)} \quad |\overline{\mathbf{u}}(t) - \mathbf{v}(t)|^2 \leq \frac{N^2}{\sigma - \gamma} \|\mathbf{u}_0(\mathbf{x}) - \overline{\mathbf{u}}(0)\|^2_{L^2(\Omega)} e^{-\gamma t} + \frac{K^2 N^2}{\gamma\sigma}.$$

Separation Norm

By a first-time argument the previous inequality holds for all $t \geq 0$. Hence

$$\|\overline{\mathbf{u}}(t) - \mathbf{v}(t)\|^2_{L^2(\Omega)} \leq \frac{N^2|\Omega|}{\sigma - \gamma}\|\mathbf{u}_0(\mathbf{x}) - \overline{\mathbf{u}}(0)\|^2_{L^2(\Omega)} e^{-\gamma t} + \frac{K^2 N^2 |\Omega|}{\gamma\sigma}$$

Separation of solutions (1)–(3) from the solutions of (4) and (5) is governed by

$$\|\mathbf{u}(\mathbf{x}, t) - \mathbf{v}(t)\|_{L^2(\Omega)} \leq A_1 \|\mathbf{u}_0(\mathbf{x}) - \mathbf{v}(0)\|_{L^2(\Omega)} e^{-\gamma t/2} + A_2$$

where A_1 and A_2 are positive constants.

Effects of Forcing

PDE:

$$\mathbf{u}_t = \mathbf{f}(\mathbf{u}) + \mathbf{g}(\mathbf{x}, t; \mathbf{p}) + D\Delta\mathbf{u} \quad \text{on } \Omega \times (0, \infty) \tag{17}$$

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}) \quad \text{on } \Omega \tag{18}$$

Spatial average:

$$\mathbf{v}_t = \mathbf{f}(\mathbf{v}) + \overline{\mathbf{g}}(t; \mathbf{p}) \tag{19}$$

$$\mathbf{v}(0) = \frac{1}{|\Omega|} \int_\Omega \mathbf{u}_0(\mathbf{x})\, d\Omega \tag{20}$$

Temporal average:

$$\mathbf{w}_t = \mathbf{f}(\mathbf{w}) + \mathbf{A} \tag{21}$$

$$\mathbf{w}(0) = \mathbf{v}(0) \tag{22}$$

Separation Result for Equilibria

Theorem 3 *If* $\mathbf{f} \in \mathcal{C}^1(\Omega)$ *where* $\Omega \subset \mathbb{R}^n$ *and* $\mathbf{e}$ *is a stable hyperbolic equilibrium of (21), and if* $\overline{\mathbf{g}} = \langle \overline{g}_1, \ldots, \overline{g}_n \rangle$ *where* $\overline{g}_i$ *is continuous and periodic in* t *with period* p_i *and has time average over a period of* A_i *for* $i = 1, \ldots, n$*, and if there exists a compact positively invariant region* $\Sigma \subset \mathbb{R}^n$ *for (17) with* $\mathbf{u}_0(\mathbf{x}) \in \Sigma$ *and if* $\mathbf{v}(t)$ *is a solution to (19) such that there exists a positive constant* γ *for which* $\langle \mathbf{f}_\mathbf{u}(\mathbf{v}(t))\mathbf{z}, \mathbf{z} \rangle \leq -\gamma |\mathbf{z}|^2$ *for all* $\mathbf{z} \in \Sigma$*, then when the smallest diffusion coefficient of* D *is sufficiently large,* $\mathbf{e}$ *is stable as a solution to (17) with homogeneous Neumann boundary conditions.*

Separation Norm

$$
\begin{aligned}
\|\mathbf{u}(\mathbf{x},t)-\mathbf{e}\|_{L^2(\Omega)} \;\le\; & A_1\|\mathbf{u}_0(\mathbf{x})-\mathbf{v}(0)\|_{L^2(\Omega)}e^{-\gamma t/2}+A_2+ \\
& B_1|\Omega|^{1/2}\sum_{i=1}^{n}p_i\|\overline{\mathbf{g}}_i\|_\infty + \\
& B_2|\Omega|^{1/2}|\mathbf{v}(0)-\mathbf{e}|e^{-\alpha t}
\end{aligned}
$$

Separation Result for Periodic Orbits

Theorem 4 *Suppose* $\mathbf{f} \in \mathcal{C}^1(\Omega)$ *where* $\Omega \subset \mathbb{R}^n$ *and* $\mathbf{w}(t)$ *is an asymptotically stable periodic solution of (21). Let the forcing term* $\overline{\mathbf{g}} = \langle \overline{g}_1, \dots, \overline{g}_n \rangle$ *where* $\overline{g}_i$ *is continuous and periodic in* t *with period* p_i *and time average over a period of* A_i *for* $i = 1, \dots, n$*. Suppose there exists a compact positively invariant region* $\Sigma \subset \mathbb{R}^n$ *for (17) with* $\mathbf{u}_0(\mathbf{x}) \in \Sigma$ *and suppose* $\mathbf{v}(t)$ *is a solution to (19) such that there exists a positive constant* γ *for which* $\langle \mathbf{f}_{\mathbf{u}}(\mathbf{v}(t))\mathbf{z}, \mathbf{z} \rangle \leq -\gamma |\mathbf{z}|^2$ *for all* $\mathbf{z} \in \Sigma$*. Then when the smallest diffusion coefficient of the diagonal diffusion matrix is sufficiently large,* $\mathbf{w}(t)$ *is stable as a solution to (17) with homogeneous Neumann boundary conditions.*

Example

$$
\begin{aligned}
u_t &= u(1 - c_{11}u - v) + g_1(x,t) + D_1 u_{xx} \\
v_t &= v(0.5 - u - c_{22}v)(u + c_{22}v - 1.5) + g_2(x,t) + D_2 v_{xx}
\end{aligned}
$$

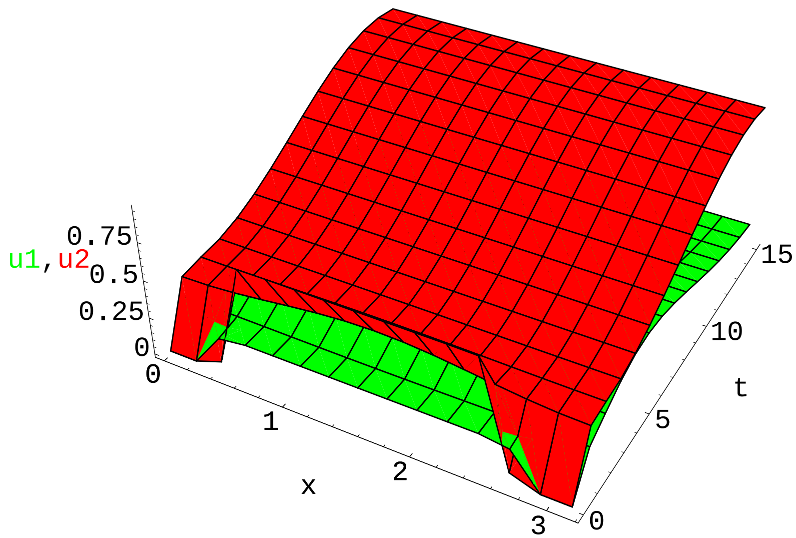
Zero-flux boundary conditions are assumed.

- $c_{11} = 1.1$, $c_{22} = 0.35$,
- $D_1 = 1$, $D_2 = 1/2$,
- and $g_1 = g_2 = 0$.

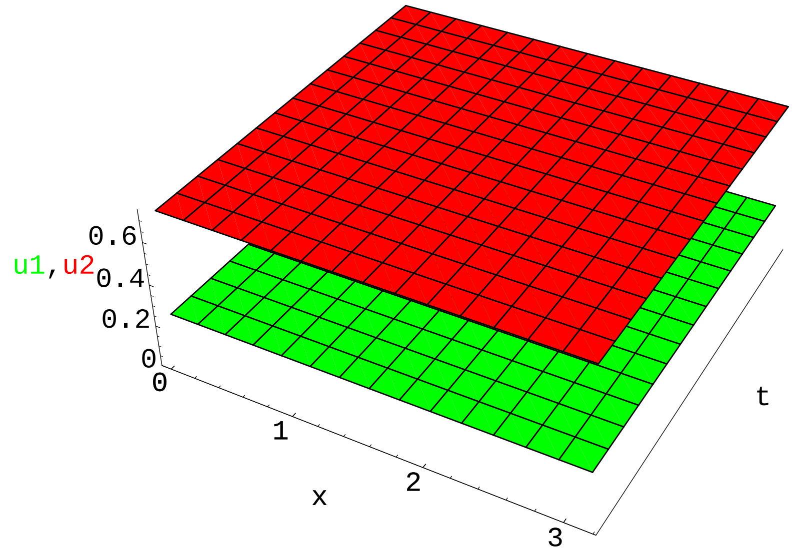
Asymtotically stable equilibrium solution

$$\mathbf{e} \approx (0.2439, 0.7317).$$

Graphs



(a) $0 \leq t \leq 15$



(b) As $t \to \infty$

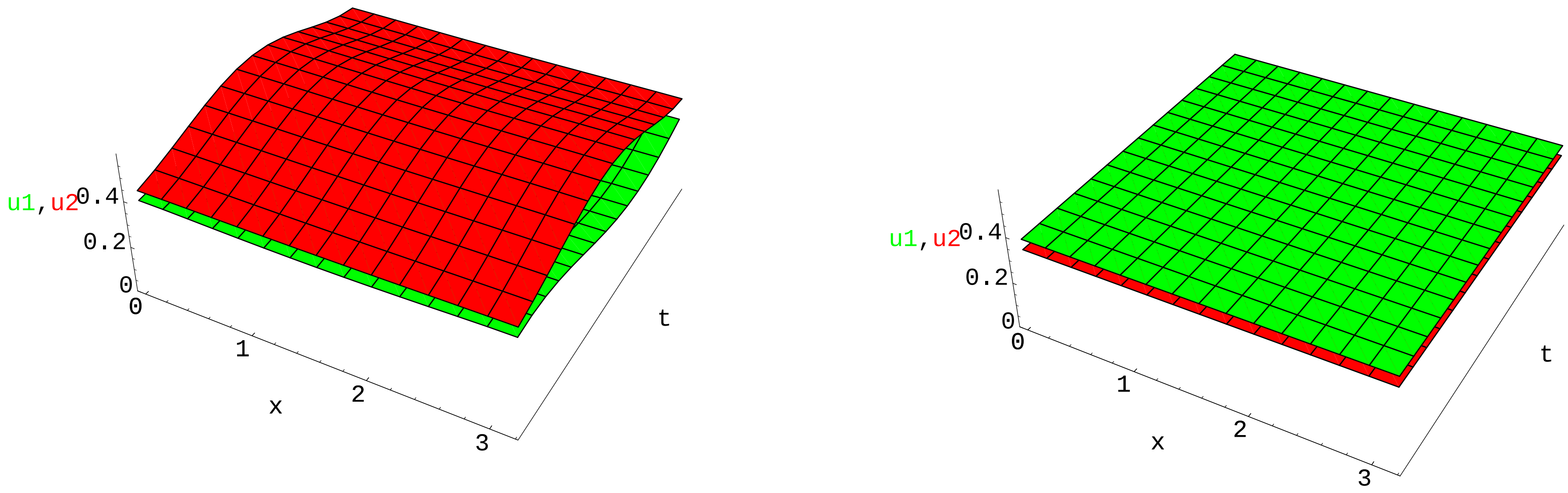
The solution to a set of reaction-diffusion equations exponentially approaches a constant equilibrium solution.

Hopf Bifurcation

As c_{11} is decreased to 1.05, the equilibrium solution undergoes a Hopf bifurcation. Nearby an attracting periodic orbit appears.
Harvesting u at the constant rate of -0.1 over all of Ω will restabilize an equilibrium at $\mathbf{e} \approx (0.3820, 0.3371)$.

The system can be brought back near the equilibrium point with time-periodic harvesting that is nonuniformly distributed in space and whose average over Ω and a period is -0.1. Let $g_1(x,t) = 0.6580B(x)(-0.1 + 0.05\cos 2\pi t)$ where $B(x)$ is a $\mathcal{C}^\infty(\mathbb{R})$ "bump function".

Graphs



(a) Spatially varying, periodic forcing (b) Constant forcing

Comparison of the asymptotic separation of solutions of system of reaction-diffusion equations subject to varying and constant forcing.