Discontinuous Forcing of Periodic Solutions in n–Dimensional $\mathcal{C}^1$ Vector Fields With Applications to Population Models

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Introduction

Compare the solutions to each of the following systems of equations.

$$\frac{dz}{dt} = F(z) \tag{1}$$

$$\frac{dx}{dt} = F(x) + G(t) \tag{2}$$

Objective: Develop a bounding result on the asymptotic separation of solutions to equation (2) from those of equation (1).

Details: "Discontinuous Forcing of Periodic Solutions in n–Dimensional $\mathcal{C}^1$ Vector Fields With Applications to Population Models," J.R. Buchanan, *Canadian Applied Mathematics Quarterly*, Volume 7, Number 4, (2000).

Assumptions

- Vector field $F : \mathbb{R}^n \to \mathbb{R}^n$ is $\mathcal{C}^1$.
- Time-dependent forcing $G : \mathbb{R} \to \mathbb{R}^n$ is piecewise continuous, periodic in t, and L^1 integrable with components having time averages of 0.
- Solution to equation (1) is periodic, nontrivial, and asymptotically stable.

Buchanan and Selgrade (1995a) established a similar result for the case of an asymptotically stable fixed point solution to (1).

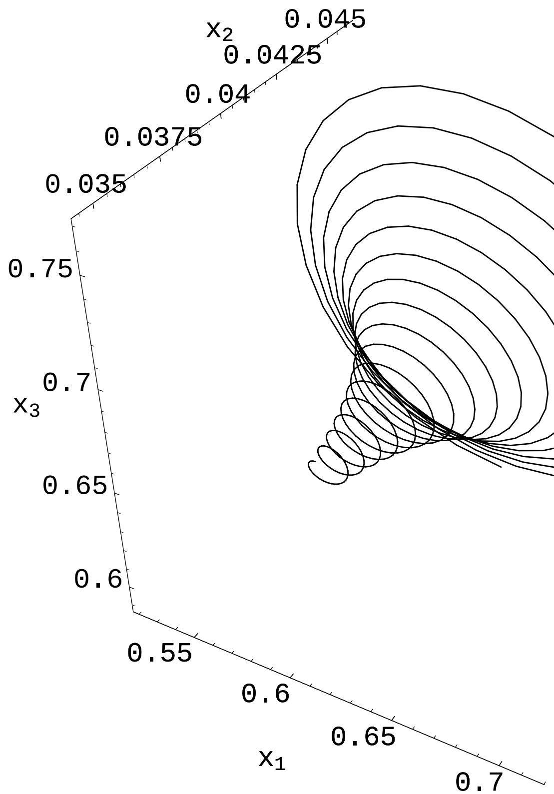
Example

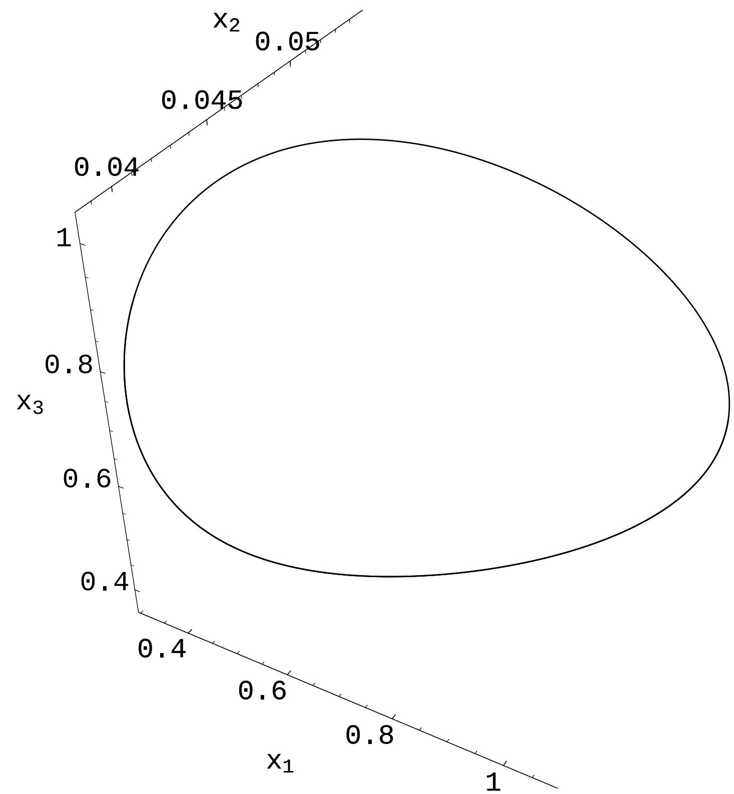
$$\frac{dx_1}{dt} = x_1(4 - 4(c_{11}x_1 + c_{22}x_2 + x_3)) + A$$

$$\frac{dx_2}{dt} = x_2(\frac{3}{4} - (c_{22}x_2 + x_3))$$

$$\frac{dx_3}{dt} = x_3(-6 + 6(x_1 + x_2 + c_{33}x_3)e^{\frac{1}{2} - \frac{1}{2}(x_1+x_2+c_{33}x_3)})$$

Buchanan and Selgrade (1995b): varying either of c_{11} or A can induce a Hopf bifurcation.





$A = -1/20$ $A = 0$

Other Parameters: $c_{11} = 0.38$, $c_{22} = 1.5$, $c_{33} = 0.5$.

Goal

Show that the normal separation of solutions to the constantly forced (1) and periodically forced (2) ODEs is related to the initial normal separation and the sum of the L^1 norms of the components of $G(t)$.

Steps:

1. Describe an n-dim moving orthonormal coord system about $\gamma(t)$.
2. Change to the new coord system in equation (2).
3. Solve the periodically forced ODE.
4. Bound the components of the solution normal to $\gamma(t)$.

Moving Orthonormal Coordinate System

Define $\Gamma = \{\gamma(\theta) \in \mathbb{R}^n | 0 \leq \theta \leq T\}$ where γ is an asymptotically stable nontrivial periodic solution of equation (1).

Unit tangent vector: $v(\theta) = \gamma'(\theta)/\|\gamma'(\theta)\|$ for $0 \leq \theta \leq T$.

Augment $v(\theta)$ with $n-1$, θ-dependent vectors to form a moving orthonormal coordinate system.

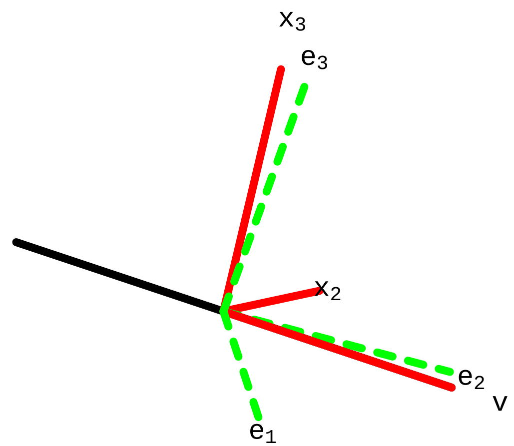
[Details in Hale (1980), pp. 214-216]

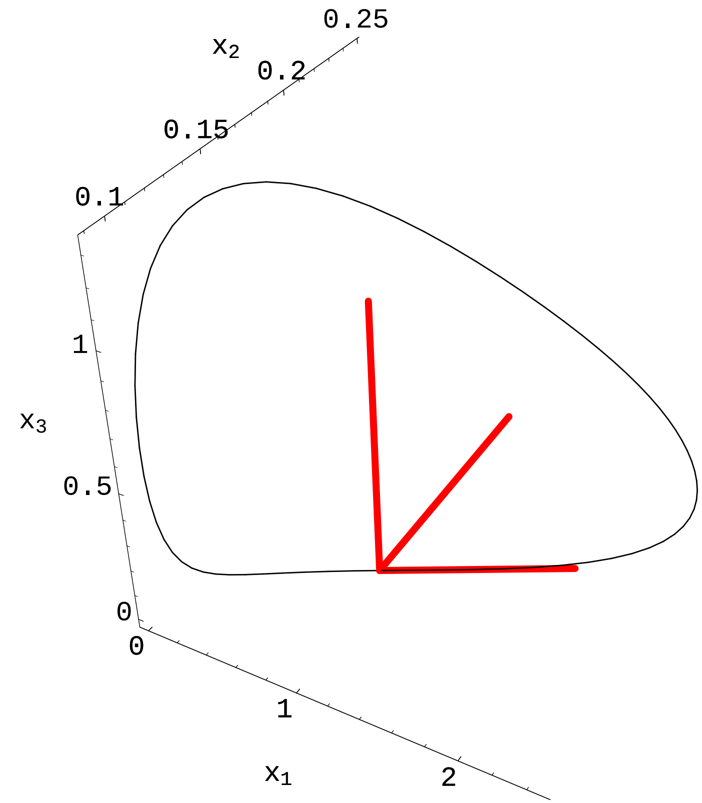
- There exists a constant unit vector $e_1 \in \mathbb{R}^n$ such that $v(\theta) \neq \pm e_1$ for all $0 \leq \theta \leq T$.
- Augment e_1 with constant vectors $\{e_2, \ldots, e_n\}$ to form an orthonormal basis for $\mathbb{R}^n$.

- Rotate each vector in the basis in the 2-dimensional plane span$\{e_1, v(\theta)\}$, until e_1 coincides with $v(\theta)$.
- Let $\{\xi_2(\theta), \ldots, \xi_n(\theta)\}$ be the rotated positions of $\{e_2, \ldots, e_n\}$.

  The set $\mathcal{B} = \{v(\theta), \xi_2(\theta), \ldots, \xi_n(\theta)\}$ is the moving orthonormal coordinate system parameterized by $0 \leq \theta \leq T$.

$$\xi_j(\theta) = e_j - \frac{e_j \cdot v(\theta)}{1 + e_1 \cdot v(\theta)}(e_1 + v(\theta)) \text{ for } j = 2, \ldots, n.$$


x3
e3
x2
e2
v
e1


0.25
x2
0.2
0.15
0.1
1
x3
0.5
0
0
1
x1
2

Change of Coordinates

Define the change of coordinates $H : \mathbb{R} \times \mathbb{R}^{n-1} \to \mathbb{R}^n$ by

$$H(\theta, \rho) = \gamma(\theta) + Z(\theta)\rho$$

where

- $\rho = (\rho_2, \ldots, \rho_n)^\top$ is the $(n-1)$ vector of normal coordinates to Γ, and
- $Z(\theta)$ is the $n \times (n-1)$ matrix whose columns are the vectors $\xi_2(\theta), \ldots, \xi_n(\theta)$.

Note that since $\xi_i(\theta)$ for $i = 2, \ldots, n$ is orthogonal to $\gamma'(\theta)$ for all θ, then $Z^\top(\theta)\gamma'(\theta) = 0$ for all θ.

Invertibility of Coordinate Transform

$H(\theta,\rho)$ is invertible provided the matrix $DH(\theta,\rho)$ is invertible.

$$DH(\theta,\rho) = \left(\begin{array}{c|c} \gamma_1'(\theta) + \sum_{i=2}^{n} \xi_{i,1}'(\theta)\rho_i & \\ \vdots & Z(\theta) \\ \gamma_n'(\theta) + \sum_{i=2}^{n} \xi_{i,n}'(\theta)\rho_i & \end{array}\right)$$

$$DH^{-1}(\theta,\rho) = \left(\frac{v(\theta)/\det DH(\theta,\rho)}{Z^\top(\theta)}\right)$$

Determinant

When $\rho = 0$

$$\det DH(\theta, 0) = \det \left(\begin{array}{c|c} \gamma_1'(\theta) & \\ \vdots & Z(\theta) \\ \gamma_n'(\theta) & \end{array} \right) = \pm \|\gamma'(\theta)\| \neq 0$$

for all $0 \leq \theta \leq T$ since Γ contains no equilibria of equation (1).

Since $\det DH(\theta, \rho)$ is continuous in ρ then there exists a constant $\beta_1 > 0$ such that if $\|\rho\| < \beta_1$ then $\det DH(\theta, \rho) \neq 0$.

New Coordinates

In (θ,ρ) coordinates equation (2) becomes

$$\begin{pmatrix} \frac{d\theta}{dt} \\ \frac{d\rho_2}{dt} \\ \vdots \\ \frac{d\rho_n}{dt} \end{pmatrix} = DH^{-1}(\theta,\rho)(F(H(\theta,\rho)) + G(t))$$

Of primary interest are the coordinates normal to Γ.

$$\begin{pmatrix} \frac{d\rho_2}{dt} \\ \vdots \\ \frac{d\rho_n}{dt} \end{pmatrix} = Z^{\top}(\theta)(F(H(\theta,\rho)) + G(t)) \tag{3}$$

Solution

Technique is that of solving a first order linear nonhomogeneous initial value problem.

Define $w(\theta, \rho) = Z^\top(\theta)F(H(\theta, \rho))$ and let

(4) $$A(\theta) = \frac{\partial w}{\partial \rho}(\theta, 0) = Z^\top(\theta)DF(\gamma(\theta))Z(\theta)$$

The $(n-1) \times (n-1)$ matrix $A(\theta)$ is a continuous and T-periodic function of θ.

Note that $w(\theta, 0) = Z^\top(\theta)F(H(\theta, 0)) = Z^\top(\theta)\gamma'(\theta) = 0$ for all θ.

Normal Variational Equation

The normal variation along Γ is given by

$$\frac{dy}{d\theta} = A(\theta)y.$$

Let $\Phi(\theta)$ be the fundamental matrix solution to the normal variational equation satisfying the initial condition,

$$\Phi(0) = I_{n-1}.$$

Subtract $A(\theta)\frac{d\theta}{dt}\rho$ from both sides of equation (3) and use the fundamental theorem of calculus.

$$
\begin{aligned}
\frac{d\rho}{dt} - A(\theta)\frac{d\theta}{dt}\rho &= Z^{\top}(\theta)(F(H(\theta,\rho)) + G(t)) - A(\theta)\frac{d\theta}{dt}\rho \\
&= w(\theta,\rho) - w(\theta,0) - A(\theta)\frac{d\theta}{dt}\rho + Z^{\top}(\theta)G(t) \\
&= \int_0^1 \left(\frac{\partial w}{\partial \rho}(\theta, s\rho) - A(\theta)\frac{d\theta}{dt}\right)\rho\, ds + Z^{\top}(\theta)G(t)
\end{aligned}
$$

where $s\rho$ for $0 \leq s \leq 1$ is a parameterization of the line joining the origin and $\rho(t)$.

Use the integrating factor: $\Phi^{-1}(\theta)$.

Solution:

$$\begin{aligned}\rho(t) &= \Phi(\theta(t))\int_0^t \Phi^{-1}(\theta)\int_0^1 \left(\frac{\partial w}{\partial \rho}(\theta, s\rho) - A(\theta)\frac{d\theta}{d\tau}\right)\rho\, ds\, d\tau \\ &\quad + \Phi(\theta(t))\int_0^t \Phi^{-1}(\theta) Z^\top(\theta) G(\tau)\, d\tau + \Phi(\theta(t))\rho(0)\end{aligned}$$

Boundedness Result

If $\gamma(t)$ is an asymptotically stable nontrivial periodic solution of (1) then for every $\epsilon > 0$, there must exist positive constants K_1, K_2, and σ and a tubular neighborhood $\mathcal{N}_\delta(\Gamma)$ around Γ of radius $0 < \delta < \epsilon$, so that if $\sum_{i=1}^{n} \|G_i\|_1$ is sufficiently small and if $x(0) \in \mathcal{N}_\delta(\Gamma)$ then $x(t) \in \mathcal{N}_\epsilon(\Gamma)$ for all $t \geq 0$ and the components ρ, of $x(t)$ normal to Γ obey the following inequality for all $t \geq 0$.

$$\|\rho(t)\| \leq K_1 \|\rho(0)\| e^{-\sigma t} + K_2 \sum_{i=1}^{n} \|G_i\|_1$$

Alternatively,

$$\|\rho(t)\| \leq K_1 \|\rho(0)\| e^{-\sigma t} + K_2 \sum_{i=1}^{n} p_i \|G_i\|_\infty.$$

Key Idea: Use integration by parts on the definite integral involving the forcing functions.

$$
\begin{aligned}
&\int_0^t \Phi(\theta(t))\Phi^{-1}(\theta(\tau))Z^\top(\theta(\tau))G(\tau)\,d\tau \\
&= \quad Z^\top(\theta(t)) \int_0^t G(\tau)\,d\tau \\
&\qquad +\Phi(\theta(t)) \int_0^t \Phi^{-1}(\theta)\left(A(\theta)Z^\top(\theta) - (Z^\top)'(\theta)\right)\int_0^\tau G(u)\,du\,\frac{d\theta}{d\tau}d\tau
\end{aligned}
$$

Note that $\frac{d}{d\theta}\Phi^{-1}(\theta) = -\Phi^{-1}(\theta)A(\theta)$.

Floquet Theory

$$\Phi(\theta) = P(\theta)e^{R\theta}$$

where $P(\theta)$ is a continuous T-periodic nonsingular matrix and R is a constant matrix.

- Since $\gamma(\theta)$ is T-periodic, the linearization of equation (1) will have a characteristic multiplier of 1.
- Since $\gamma(\theta)$ is asymptotically stable, the remaining characteristic multipliers $\mu_2, \ldots, \mu_n$ have moduli less than 1.
- $\mu_2, \ldots, \mu_n$ are the characteristic multipliers of equation (4) [Hale (1980)].

There exist positive real numbers λ and K such that

$$\|e^{R\theta}\| \leq K e^{-\lambda\theta} \qquad \text{for } \theta \geq 0.$$

There exist positive constants M_0, M_1, M_2 and M_3 such that

$$\begin{aligned}
\max_{\theta\geq 0} \|Z^\top(\theta)\| &\leq M_0 \\
\max_{\theta\geq 0} \|P(\theta)\| &\leq M_1 \\
\max_{\theta\geq 0} \|P^{-1}(\theta)\| &\leq M_2 \\
\max_{\theta\geq 0} \|A(\theta)Z^\top(\theta) - (Z^\top)'(\theta)\| &\leq M_3
\end{aligned}$$

The definite integral involving the forcing functions is bounded like

$$\begin{aligned}\left\|\int_0^t \Phi(\theta(t))\Phi^{-1}(\theta(\tau))Z^\top(\theta(\tau))G(\tau)\,d\tau\right\| \\ \leq \quad \left(M_0 + \frac{KM_1M_2M_3}{\lambda}(1-e^{-\lambda\theta(t)})\right)\sum_{i=1}^n \|G_i\|_1\end{aligned}$$

Applying the triangle inequality and the previous results we have

$$\begin{aligned}\|\rho(t)\| \quad \leq \quad & KM_1M_2e^{-\lambda\theta(t)}\int_0^t e^{\lambda\theta}\int_0^1 \left\|\frac{\partial w}{\partial\rho}(\theta, s\rho) - A(\theta)\theta'\right\| \|\rho\|\,ds\,d\tau \\ & + \left(M_0 + \frac{KM_1M_2M_3}{\lambda}(1-e^{-\lambda\theta(t)})\right)\sum_{i=1}^n \|G_i\|_1 \\ & + KM_1\|\rho(0)\|e^{-\lambda\theta(t)}\end{aligned}$$

Assume the expression,

$$\left\| \frac{\partial w}{\partial \rho}(\theta, s\rho) - A(\theta)\frac{d\theta}{d\tau} \right\|$$

can be bounded by a constant L to be determined later, then

$$\begin{aligned} \|\rho(t)\| \leq\ & KM_1 e^{-\lambda\theta(t)} \left(LM_2 \int_0^t e^{\lambda\theta} \|\rho(\tau)\|\, d\tau + \|\rho(0)\| \right) \\ & + \left(M_0 + \frac{KM_1M_2M_3}{\lambda}(1 - e^{-\lambda\theta(t)}) \right) \sum_{i=1}^{n} \|G_i\|_1 \end{aligned}$$

After multiplying both sides by $e^{\lambda\theta(t)}$ and applying Gronwall's inequality [Miller and Michel (1982)] we obtain an explicit inequality describing $\|\rho(t)\|$.

Let

$$KM_1 = B_1, \quad KM_1M_2 = B_2, \text{ and } \frac{KM_1M_2M_3}{\lambda} = B_3,$$

then

$$\begin{aligned} &\|\rho(t)\| \\ &\leq \quad (M_0 + B_3)\left(1 + B_2L\int_0^t e^{B_2L(t-\tau)-\lambda(\theta(t)-\theta(\tau))}\,d\tau\right)\sum_{i=1}^{n}\|G_i\|_1 \\ &\quad +e^{B_2Lt-\lambda\theta(t)}\left(B_1\|\rho(0)\| + B_3\sum_{i=1}^{n}\|G_i\|_1\right) \end{aligned}$$

When ρ and $G(t)$ are small

$$\begin{aligned} \frac{d\theta}{dt} &\approx \frac{v(\theta)}{\det DH(\theta,0)}(F(H(\theta,0))) \\ &= \frac{v(\theta)\cdot F(\gamma(\theta))}{\|\gamma'(\theta)\|} \\ &= 1 \end{aligned}$$

By the continuous dependence of $w(\theta,\rho)$ on ρ and the definition of $A(\theta)$, then the expression

$$\left\| \frac{\partial w}{\partial \rho}(\theta, s\rho) - A(\theta)\frac{d\theta}{d\tau} \right\|$$

can be bounded as small as desired by choosing an appropriately small bound on ρ.

For each $\alpha \in (0,1)$ there exist bounds on $\|\rho\|$ and $\|G_i\|_\infty$ for $i = 1, \ldots n$ such that $|\frac{d\theta}{dt} - 1| < \alpha$. This implies

$$B_2 L(t-\tau) - \lambda(\theta(t) - \theta(\tau)) < B_2 L(t-\tau) - \lambda(1-\alpha)(t-\tau).$$

Thus if we bound $\|\rho(t)\|$ sufficiently small, say by $0 < \beta \leq \beta_1$, then L can be made small enough that $B_2 L - \lambda(1-\alpha) = -\sigma < 0$.

$$\|\rho(t)\| \leq B_1 e^{-\sigma t}\|\rho(0)\| + \left[(B_3 + M_0)\left(1 + \frac{B_2 L}{\sigma}\right) + B_3\right] \sum_{i=1}^{n} \|G_i\|_1$$

Let $K_1 = B_1$ and $K_2 = \left[(B_3 + M_0)\left(1 + \frac{B_2 L}{\sigma}\right) + B_3\right]$

A priori Assumptions

Insure that $\|\rho(t)\| < \beta$ for all $t \geq 0$ so that the *a priori* assumptions made earlier are never violated.

- Suppose that $\|\rho_0\| < \min\{\frac{\beta}{3}, \frac{\beta}{3K_1}\}$ and
  $$\sum_{i=1}^{n} p_i \|G_i\|_\infty < \min\{\frac{\beta}{3}, \frac{\beta}{3K_2}\}$$
- If $\|\rho(\hat{t})\| \geq \beta$ for some $\hat{t} \geq 0$, assume that $\hat{t}$ is the first such time.
- For all $0 \leq t < \hat{t}$ we have $\|\rho(t)\| < \beta$. Hence for all $t < \hat{t}$ we have $\sigma > 0$ and thus

$$\|\rho(t)\| \leq K_1 \frac{\beta}{3K_1} e^{-\sigma t} + K_2 \frac{\beta}{3K_2} \leq \frac{\beta}{3} + \frac{\beta}{3} = \frac{2}{3}\beta < \beta.$$

An Example

Three species model

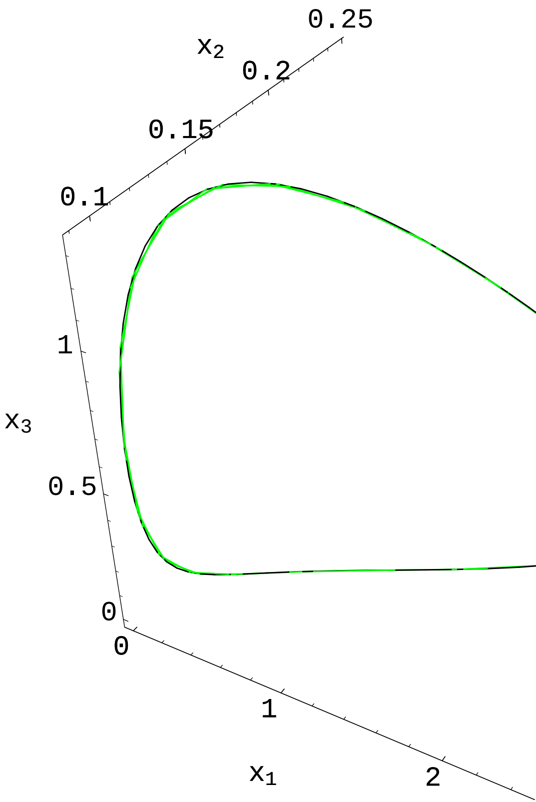
$$\frac{dx_1}{dt} = x_1(4 - 4(c_{11}x_1 + c_{22}x_2 + x_3)) + A + C\sin\frac{2\pi}{B}t$$

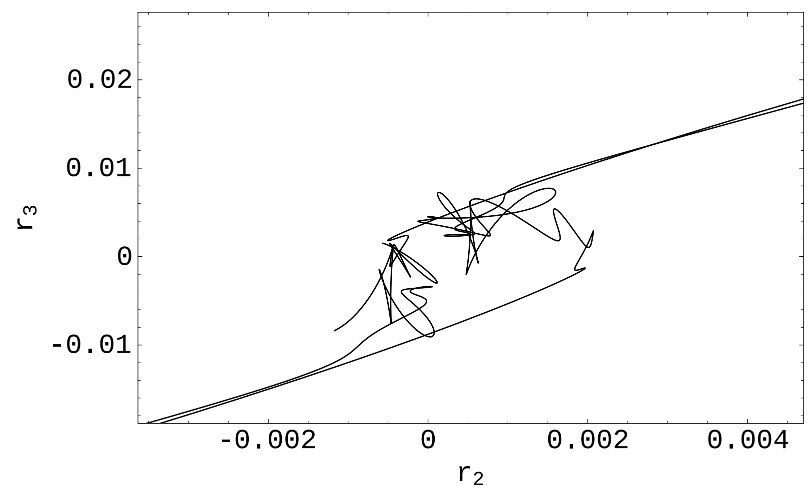
$$\frac{dx_2}{dt} = x_2(\frac{3}{4} - (c_{22}x_2 + x_3))$$

$$\frac{dx_3}{dt} = x_3(-6 + 6(x_1 + x_2 + c_{33}x_3)e^{\frac{1}{2}-\frac{1}{2}(x_1+x_2+c_{33}x_3)})$$

In each of the following cases $c_{11} = 0.21$, $c_{22} = 1.5$, $c_{33} = 0.5$, and $A = -0.02$.

Small Amplitude Forcing

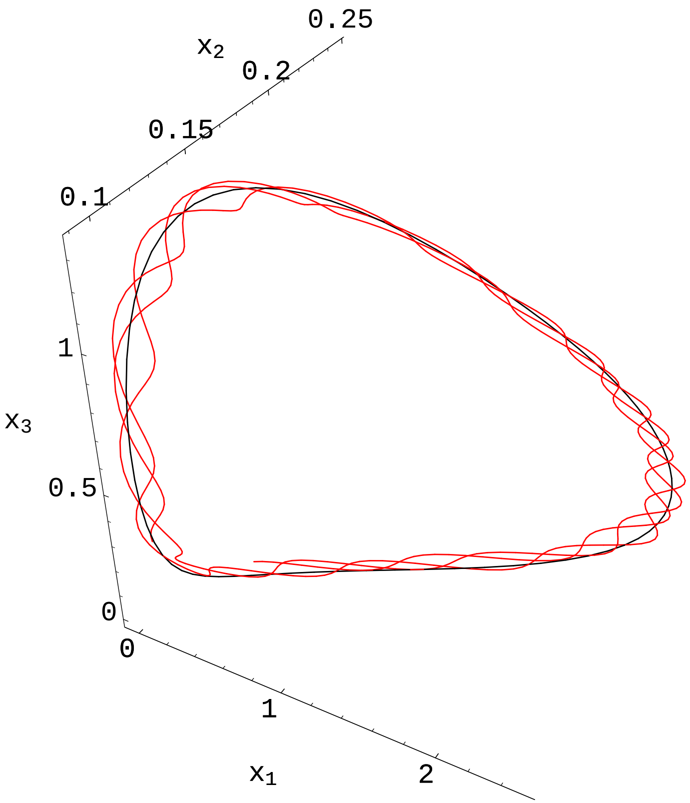


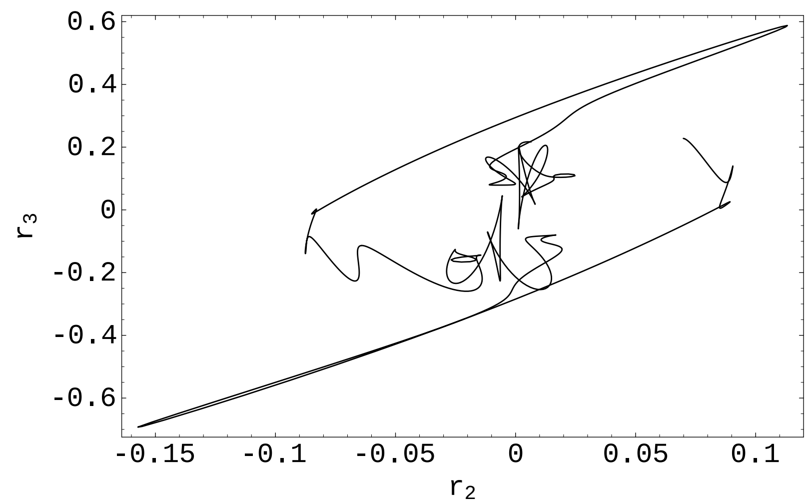


Parameters: $B = 0.5$, $C = 0.05$.

Initial conditions: $(x_1, x_2, x_3) = (1.981696, 0.177987, 0.086774)$

Large Amplitude Forcing





Parameters: $B = 0.5$, $C = 1.5$.

Initial conditions: $(x_1, x_2, x_3) = (0.379027, 0.128368, 0.066091)$

Separation Surface