Aspects of Financial Mathematics:

Options, Derivatives, Arbitrage, and the Black-Scholes Pricing Formula

J. Robert Buchanan

Millersville University of Pennsylvania

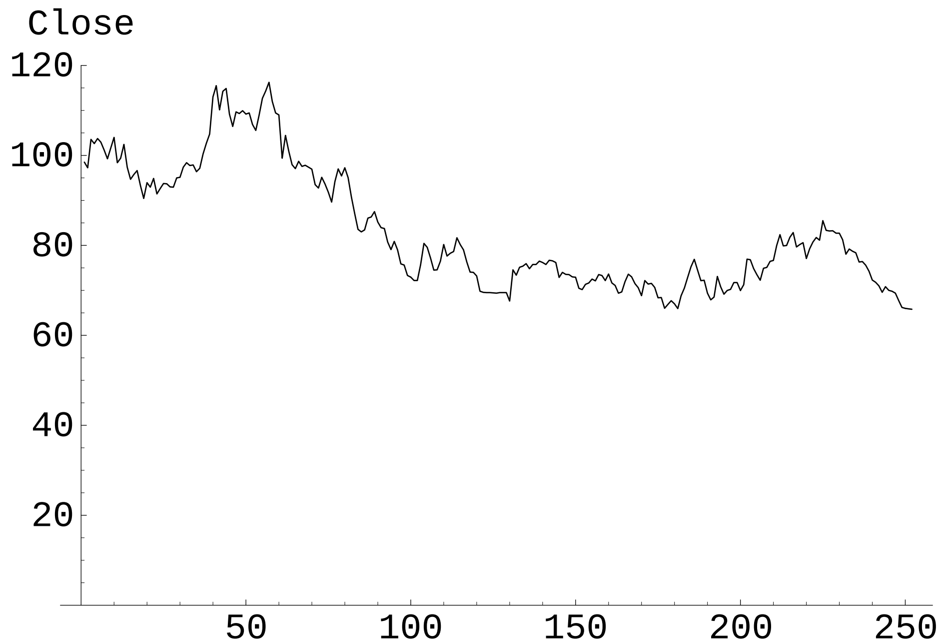
email: Bob.Buchanan@millersville.edu

Terminology

- **Arbitrage**: a trading strategy which takes advantage of two securities being mispriced relative to one another in order to make a profit.
- **Options**: the right, but not the obligation, to purchase or sell a security at an agreed upon price in the future.
- **Volatility**: the range of movement in the price of a security
- **Black-Scholes Pricing Formula**: a mathematical formula developed by Fischer Black and Myron Scholes (and extended by Robert Merton) for pricing options. They won the Nobel Prize in Economics in 1997 for this work.

Why Study Financial Mathematics?

To reduce the risks inherent in investing.



Closing prices of Sony Corporation stock traded on the NYSE between 6/23/2000 and 7/03/2001. Data obtained from `http://www.financialweb.com/`.

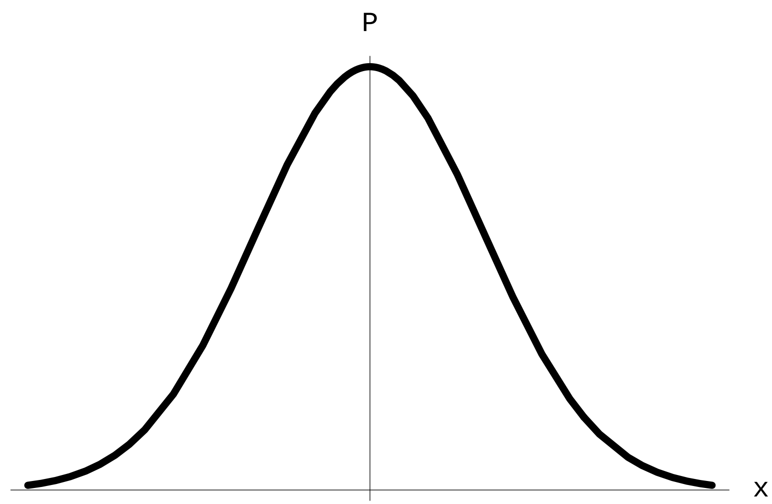
Efficient Market Hypothesis

- The present price of a security reflects the entire past history of the security.
- The past history holds no additional information.
- The price of the security responds immediately to new information.

The **relative change** in the price of a security is more important than the **absolute change.**

Lognormal Random Variables

- **Random variable**: a quantity characterized as being able to take on different values with different probabilities.
- **Normal distribution**: a formula giving the probability of a random variable having a "bell-shaped" distribution.



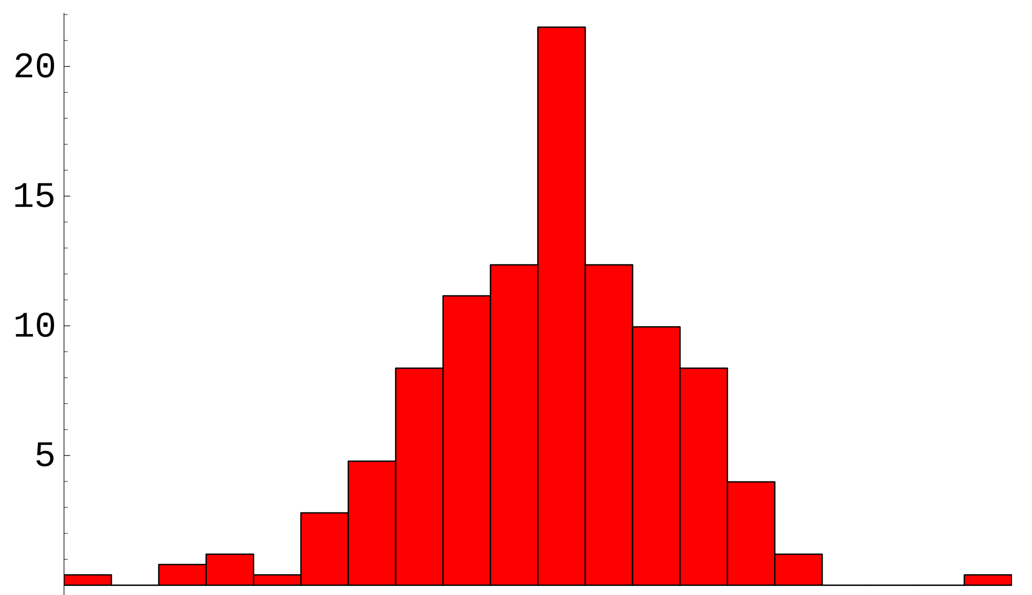
- **Lognormal distribution**: a formula giving the probability of a random variable whose logarithm has a normal distribution.

Lognormal Changes in Sony Stock

Starting with the closing prices $\{S(0), S(1), \ldots, S(252)\}$, form the random variable

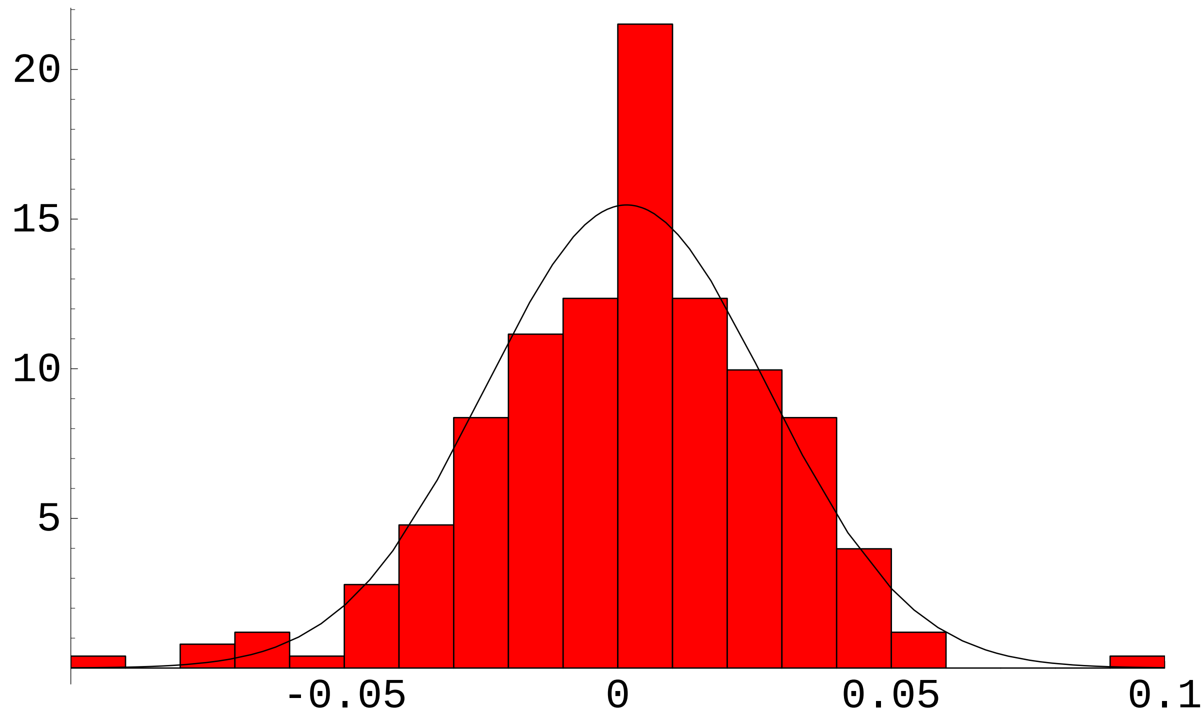
$$X(n) = \ln\left(\frac{S(n+1)}{S(n)}\right),$$

which appears to be normally distributed.



Sony Statistics

- Expected value, $\mu \approx 0.00160732$.
- Standard deviation or volatility, $\sigma \approx 0.0257846$.



Stochastic Models

- **Model of risk-free investing:** continuously compounded interest,

$$S(t) = S_0 e^{\mu t}.$$

In this case

$$d(\ln S(t)) = \ln\left(\frac{S(t+dt)}{S(t)}\right) = \ln\left(\frac{S_0 e^{\mu(t+dt)}}{S_0 e^{\mu t}}\right) = \mu\, dt.$$

- **Model incorporating unexpected news:** geometric Brownian motion,

$$d(\ln S(t)) = \mu\, dt + \sigma \sqrt{dt}\, dz$$

where z is a standard normal random variable.

Properties of $d(\ln S(t))$

$$\begin{aligned}
\mathsf{E}[d(\ln S(t))] &= \mathsf{E}[\mu\, dt + \sigma\sqrt{dt}\, dz] \\
&= \mu\, dt + \sigma\sqrt{dt}\,\mathsf{E}[dz] \\
&= \mu\, dt
\end{aligned}$$

$$\begin{aligned}
\mathsf{Var}(d(\ln S(t))) &= \mathsf{E}[d(\ln S(t))^2] - \mathsf{E}[d(\ln S(t))]^2 \\
&= (\mu\, dt)^2 + \sigma^2\, dt\,\mathsf{E}[(dz)^2] - (\mu\, dt)^2 \\
&= \sigma^2\, dt\,\mathsf{Var}(dz) \\
&= \sigma^2\, dt
\end{aligned}$$

which explains why the volatility scales like $\sqrt{dt}$.

Change of Variables

A more natural quantity than $d(\ln S)$ to model is dS. In Calculus I we used to learn that

$$d(\ln S) = \frac{dS}{S},$$

so wouldn't

$$d(\ln S) = \mu\, dt + \sigma \sqrt{dt}\, dz$$

imply

$$dS = \mu S\, dt + \sigma S \sqrt{dt}\, dz?$$

Actually, no.

Itô's Lemma

Suppose random process x is defined by the stochastic differential equation

$$dx = a(x,t)\,dt + b(x,t)\,dz,$$

where z is a normal random variable and suppose $y = F(x,t)$, then

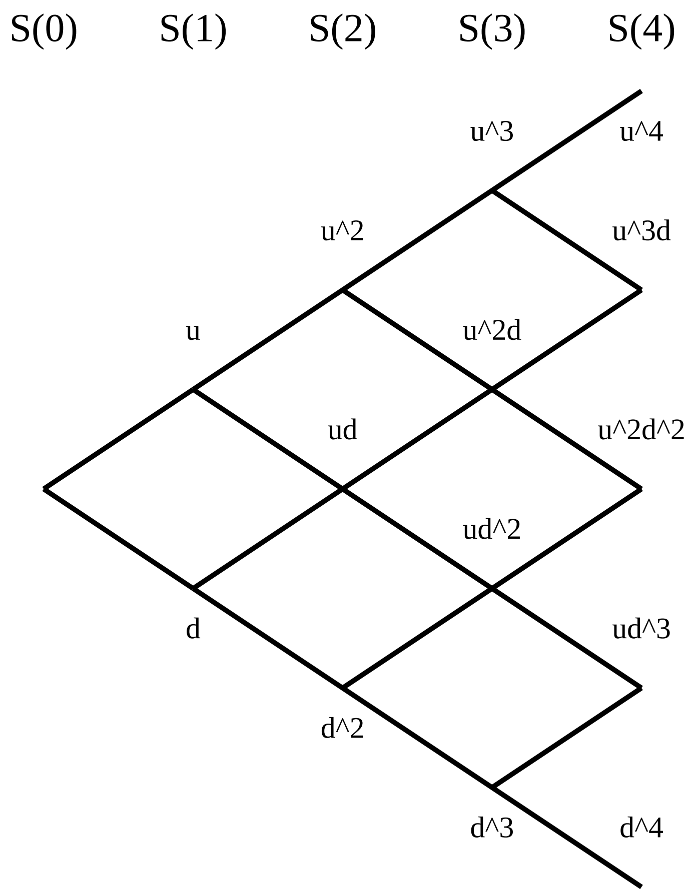
$$dy = \left[a\frac{\partial F}{\partial x} + \frac{\partial F}{\partial t} + \frac{1}{2}b^2\frac{\partial^2 F}{\partial x^2}\right]dt + b\frac{\partial F}{\partial x}\,dz.$$

Thus

$$dS = \left(\mu + \frac{1}{2}\sigma^2\right)S\,dt + \sigma S\sqrt{dt}\,dz$$

Binomial Lattice Model

Assumptions: Price of a security can only go up by a factor $u > 1$ with probability $0 < p < 1$ or down by a factor $0 < d < 1$ with probability $1 - p$.



Lattice Parameters

For a single time step of size dt,

$$\begin{aligned} \mu\, dt &= p \ln u + (1-p) \ln d \\ \sigma^2\, dt &= p(\ln u)^2 + (1-p)(\ln d)^2 - (p \ln u + (1-p) \ln d)^2. \end{aligned}$$

Assume that $d = 1/u$ and derive the system of two equations and two unknowns,

$$\begin{aligned} \mu\, dt &= (2p-1) \ln u \\ \sigma^2\, dt &= 4p(1-p)(\ln u)^2. \end{aligned}$$

Square the first equation and add to the second.

u, d, p, and all that

Thus we have,

$$\begin{aligned} \ln u &= \sqrt{\mu^2 (dt)^2 + \sigma^2 dt} \\ 2p - 1 &= \frac{\mu\, dt}{\sqrt{\mu^2 (dt)^2 + \sigma^2 dt}} \end{aligned}$$

Assume that dt is small and finally we have the approximations,

$$u \approx e^{\sigma\sqrt{dt}}, \quad d \approx e^{-\sigma\sqrt{dt}}, \quad p \approx \frac{1}{2}\left(1 + \frac{\mu}{\sigma}\sqrt{dt}\right).$$

The volatility affects the relative change in the value of the security, not the drift parameter.

Sony Parameters

For the Sony Corp. data shown earlier,

$$\begin{aligned} u &\approx 1.02612 \\ d &\approx 0.974545 \\ p &\approx 0.531168 \end{aligned}$$

To model future values of the security take a random walk through the binomial lattice using these parameters or use the discrete version of the stochastic process.

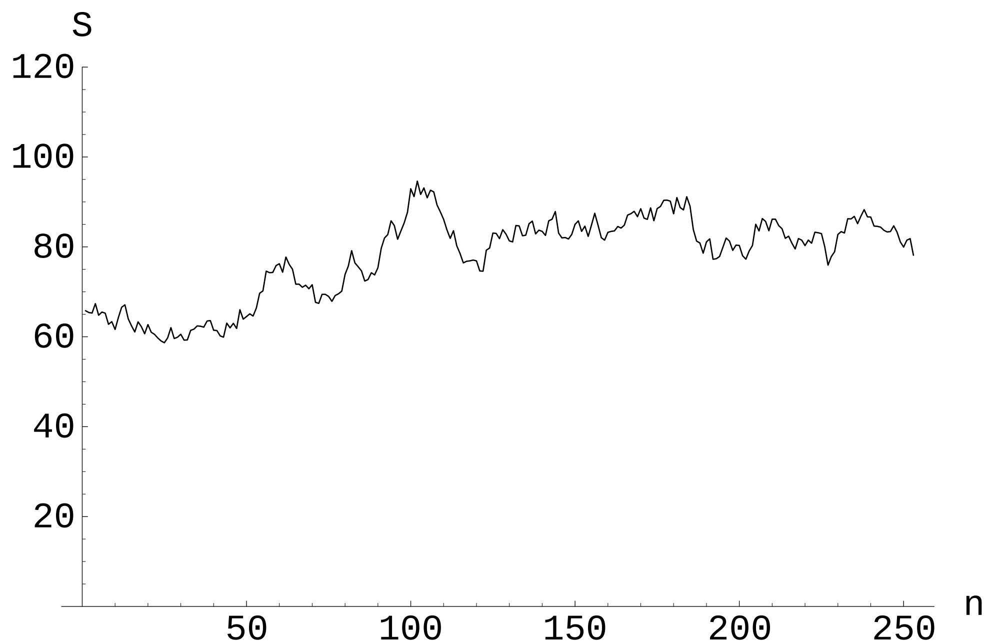
$$\ln S(t+\Delta t) - \ln S(t) = \mu\Delta t + \sigma\sqrt{\Delta t}z(t)$$

leads to

$$S(t+\Delta t) = S(t)e^{\mu\Delta t+\sigma\sqrt{\Delta t}z(t)}.$$

Realization

Using either approach we could obtain this realization of the future values of the security.



Options and Arbitrage

- **Call**: an option which allows the owner to buy a security in the future at a guaranteed price. The symbol C will denote the price of a call option.
- **Put**: an option (with price P) which allows the owner to sell a security in the future at a guaranteed price.
- **Strike price**: the future guaranteed price (K) of the security for the owner of an option.
- **Expiration time**: the future date (T) by which an option must be exercised or it is lost.
  - **European options**: exercised only when $t = T$.
  - **American options**: exercised whenever $0 \leq t \leq T$.

European Put-Call Parity

There exists a relationship between the price of a security S, the prices of calls C and puts P with the same strike price K and exercise time T, and the prevailing risk-free interest rate r.

$$S + P_e = C_e + Ke^{-rT}$$

If this relationship does not hold, then there is a risk-free way to make a guaranteed profit with no personal investment.

The following two examples suggest a means by which this formula is proven.

Example 1

Suppose $S + P_e > C_e + Ke^{-rT}$.
Let $S = 31$, $K = 30$, $C_e = 3$, $P_e = 2.25$, r = 10%, and $T = 0.25$. Then

$$\begin{aligned} S + P_e &= 33.25 \\ C_e + Ke^{-rT} &= 32.26 \end{aligned}$$

1. Buy the Call and sell short the security and the Put. This would generate in cash $S + P_e - C_e = 30.25$.
2. Invest our cash for the life of the option in the bank. After 3 months we have 31.02 in the bank.
3. At the exercise time we buy the security at the strike price and walk away with a profit of $31.02 - 30 = 1.02$.

Example 2

Suppose $S + P_e < C_e + Ke^{-rT}$.
Let $S = 31$, $K = 30$, $C_e = 3$, $P_e = 1$, r = 10%, and $T = 0.25$.
Then

$$
\begin{aligned}
S + P_e &= 32 \\
C_e + Ke^{-rT} &= 32.26
\end{aligned}
$$

1. Buy the security and the Put and sell short the Call. This would require that we borrow $S + P_e - C_e = 29$.
2. After 3 months we owe the bank 29.73.
3. At the exercise time we sell the security at the strike price and walk away with a profit of $30 - 29.73 = 0.27$.

How do you price a European option?

We will assume the underlying security

- follows the lognormal random walk described earlier,
- pays no dividends,
- there are no transaction costs in trading the security or the option.

There are at least two essentially equivalent ways to determine the price of an option:

- Derive and solve a partial differential equation,
- Use the binomial lattice with a small Δt.

Binomial Lattice Approach

Assumptions:

- The risk-free interest rate for both borrowing and lending is r.
- European call option expires n periods from now.
- There is no arbitrage, *i.e.* there is no guaranteed profit from buying or selling the security or the option.

Value of security: $S(t+n\Delta t) = u^Y d^{n-Y} S(t)$

Value of option: $\max\{S(t+n\Delta t) - K, 0\} = (S(t+n\Delta t) - K)^+$

Present Value

Since the option must be priced at time t, then its present value is

$$(1 + r\Delta t)^{-n}(S(t + n\Delta t) - K)^+,$$

and thus the expected value of the call option is

$$C = (1 + r\Delta t)^{-n}\mathsf{E}[(u^Y d^{n-Y} S(t) - K)^+].$$

Note that in an arbitrage-free setting the probability of taking a particular branch in the binomial lattice is affected by r. The expected gain from purchasing the security at time t is

$$\begin{aligned} 0 &= \frac{pu}{1 + r\Delta t} S(t) + \frac{(1-p)d}{1 + r\Delta t} S(t) - S(t), \\ \Longrightarrow \qquad p &= \frac{1 + r\Delta t - d}{u - d}. \end{aligned}$$

Example Call Pricing

$S = 100$, $r = 0.06/12$, $n = 6$, $K = 101$, $\mu = 0.12/12$, $\sigma = 0.20/12$

Security price lattice:

						110.5
					108.69	
				106.894		106.8
			105.127		105.127	
		103.39		103.39		103.
	101.681		101.681		101.681	
100		100		100		10
	98.3471		98.3471		98.3471	
		96.7216		96.7216		96.72
			95.1229		95.1229	
				93.5507		93.55
					92.0044	
						90.48

						0.0725
					0.112352	
				0.173966		0.238
			0.269369		0.30807	
		0.417092		0.381613		0.327
	0.645826		0.443169		0.337893	
1		0.457469		0.313917		0.239
	0.354174		0.243035		0.185302	
		0.125439		0.114769		0.0984
			0.0444271		0.0508101	
				0.0157349		0.0215
					0.00557289	
						0.0019

Example Call Pricing II

						9.517
					7.6904	
				5.89391		5.893
			4.12711		4.12711	
		2.38951		2.38951		2.389
	0.680633		0.680633		0.680633	
-1		-1		-1		-1
	-2.65285		-2.65285		-2.65285	
		-4.27839		-4.27839		-4.27
			-5.87706		-5.87706	
				-7.4493		-7.44
					-8.99556	
						-10.5

						9.517
					7.6904	
				5.89391		5.893
			4.12711		4.12711	
		2.38951		2.38951		2.389
	0.680633		0.680633		0.680633	
0		0		0		0
	0		0		0	
		0		0		0
			0		0	
				0		0
					0	
						0

$p \approx 0.64583$, $u \approx 1.01681$, $d \approx 0.98347$ which implies that $C \approx 2.79499$.

Black-Scholes Formula

The Black-Scholes Formula is derived by passing to the limit as $\Delta t \to 0$ and using the Central Limit Theorem. The price of a European Call is

$$\begin{aligned} C &= S\phi(w) - Ke^{-r(T-t)}\phi(w - \sigma\sqrt{T-t}), \\ \text{where} \quad w &= \frac{1}{\sigma\sqrt{T-t}}\left[(r + \frac{\sigma^2}{2})(T-t) - \ln(K/S)\right], \\ \text{and} \quad \phi(w) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{w} e^{-x^2/2}\,dx. \end{aligned}$$

Note: the European Call option price of the previous example would be $C \approx 2.7955$.

Price of a Put

Using the Put-Call Parity Formula and the Black-Scholes Formula together, the formula for the price of a Put should be

$$P = S(\phi(w) - 1) - Ke^{-r(T-t)}(\phi(w - \sigma\sqrt{T-t}) - 1).$$

Note: The prices of options do not depend on knowledge of whether the price of the security is likely to go up or down.

Partial Differential Equation Approach

Stochastic process governing S:

$$dS = (\mu + \sigma^2/2)S\,dt + \sigma S\sqrt{dt}\,dz$$

Let $F(S,t)$ be the value of a financial derivative. Apply Itô's Lemma.
Stochastic process governing F:

$$dF = \left((\mu + \sigma^2/2)S\frac{\partial F}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} + \frac{\partial F}{\partial t}\right)\,dt + \sigma S\frac{\partial F}{\partial S}\sqrt{dt}\,dz.$$

Eliminate the stochastic part. Create a portfolio consisting of the security and the derivative.

$$P = F - \Delta S$$

Portfolio

$-\Delta$ is a fractional number of units of the security in the portfolio.
Stochastic process governing the portfolio:

$$\begin{aligned} dP &= dF - \Delta\, dS \\ &= \left[(\mu + \sigma^2/2)S\left(\frac{\partial F}{\partial S} - \Delta\right) + \frac{\sigma^2 S^2}{2}\frac{\partial^2 F}{\partial S^2} + \frac{\partial F}{\partial t}\right] dt \\ &\quad + \sigma S\left(\frac{\partial F}{\partial S} - \Delta\right)\sqrt{dt}\, dz \end{aligned}$$

Choose $\Delta = \partial F/\partial S$ and obtain

$$dP = \left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} + \frac{\partial F}{\partial t}\right) dt.$$

Arbitrage-free Assumption

1. Invest P in a risk-free bond at interest rate r, *or*
2. Invest P in the portfolio.

Difference in returns should be

$$
\begin{aligned}
0 &= rP\,dt - dP \\
\Longrightarrow \qquad rP\,dt &= \left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} + \frac{\partial F}{\partial t}\right) dt \\
\Longrightarrow \qquad rF &= \frac{\partial F}{\partial t} + rS\frac{\partial F}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2}
\end{aligned}
$$

Black-Scholes PDE

Amazingly the linear **Black-Scholes PDE** prices every possible type of financial derivative. The only difference being the **boundary** and **final conditions** we impose.

If $F(S,t)$ is a European call option, then

- Boundary conditions: $F(0,t) = 0$ and $F(S,t) \to S$ as $S \to \infty$
- Final condition: $F(S,T) = (S-K)^+$

If $F(S,t)$ is a European put option, then

- Boundary conditions: $F(0,t) = Ke^{-r(T-t)}$ and $F(S,t) \to 0$ as $S \to \infty$
- Final condition: $F(S,T) = (K-S)^+$

Change of Variables I

Through an appropriate change of variables, the Black-Scholes PDE can be converted to the **Heat Equation.**

Let $x = \ln(S/K), \quad \tau = \frac{1}{2}\sigma^2(T-t), \quad F(S,t) = Kv(x,\tau)$.

Then $\frac{\partial F}{\partial t} = -\frac{K\sigma^2}{2}\frac{\partial v}{\partial \tau}, \quad \frac{\partial F}{\partial S} = e^{-x}\frac{\partial v}{\partial x}, \quad \frac{\partial^2 F}{\partial S^2} = \frac{1}{K}e^{-2x}\left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x}\right).$

Substituting in the Black-Scholes equation produces

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k-1)\frac{\partial v}{\partial x} - kv$$

where $k = 2r/\sigma^2$.

Change of Variables II

If $F(S,t)$ describes a European call option, then the final condition becomes an initial condition since

$$F(S,T) = (S-K)^+ \quad \Longleftrightarrow \quad v(x,0) = (e^x - 1)^+.$$

Another change of variables: let α and β be constants and

$$v(x,\tau) = e^{\alpha x + \beta \tau} u(x,\tau),$$

then

$$\begin{aligned} v_\tau(x,\tau) &= e^{\alpha x+\beta\tau}(\beta u(x,\tau) + u_\tau(x,\tau)) \\ v_x(x,\tau) &= e^{\alpha x+\beta\tau}(\alpha u(x,\tau) + u_x(x,\tau)) \\ v_{xx}(x,\tau) &= e^{\alpha x+\beta\tau}(\alpha^2 u(x,\tau) + 2\alpha u_x(x,\tau) + u_{xx}(x,\tau)) \end{aligned}$$

Change of Variables III

Substitute into the previous PDE and we obtain,

$$u_\tau = u_{xx} + (2\alpha + k - 1)u_x + (\alpha^2 - \beta + (k-1)\alpha - k)u.$$

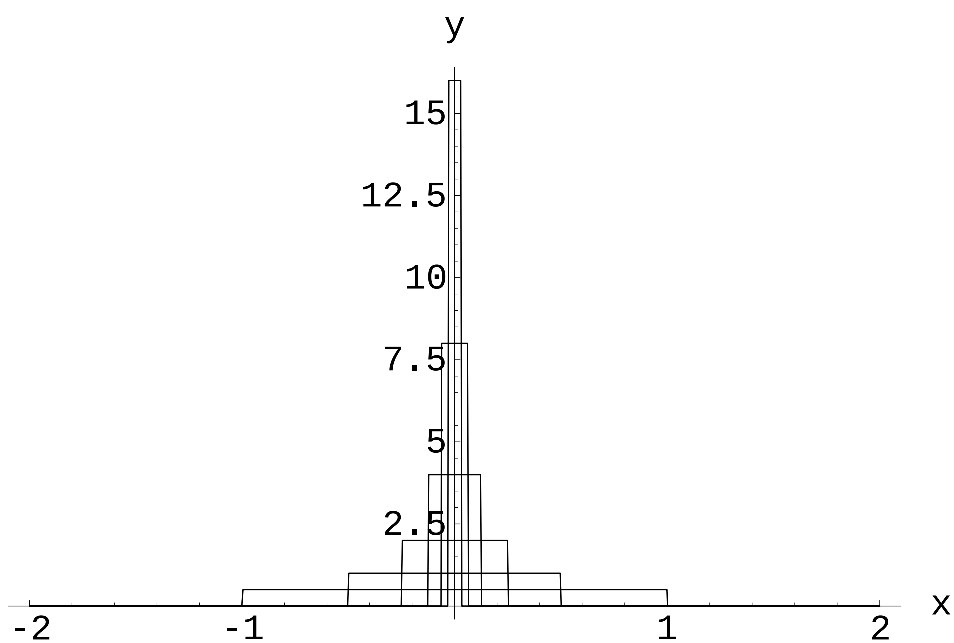
Let $\alpha = (1-k)/2$ and $\beta = -(1+k)^2/4$ and we have the Heat Equation

$$\begin{aligned} u_\tau &= u_{xx} \\ \text{(IC)} \qquad u(x,0) &= (e^{(k+1)x/2} - e^{(k-1)x/2})^+ \end{aligned}$$

Dirac Delta Function

$\delta(x)$ is not a function in the ordinary sense, but belongs to a class of “generalized functions”.

$$\delta(x) = \lim_{\epsilon \to 0} \begin{cases} \dfrac{1}{2\epsilon} & \text{if } -\epsilon < x < \epsilon, \\ 0 & \text{otherwise.} \end{cases}$$



Properties of $\delta(x)$

1. $\delta(x) = 0$ for all $x \neq 0$.
2. $\displaystyle\int_{-\infty}^{\infty} \delta(x)\, dx = 1$
3. If $f(x)$ is continuous at $x = 0$ then
$$\int_{-\infty}^{\infty} \delta(x) f(x)\, dx = f(0)$$

Fundamental Solution

Initial value problem:

$$
\begin{aligned}
u_\tau &= u_{xx} && \text{for } -\infty < x < \infty,\ \tau > 0 \\
u(x,0) &= \delta(x) && \text{for } -\infty < x < \infty \\
\lim_{|x|\to\infty} u(x,\tau) &= 0 && \text{for } \tau > 0
\end{aligned}
$$

Let $z = x/\sqrt{\tau}$ and suppose $u(x,\tau) = \tau^{-1/2}V(z)$.

$$
\begin{aligned}
u_\tau &= -\frac{1}{2}\tau^{-3/2}\left(V(z) + zV'(z)\right) \\
u_{xx} &= \tau^{-3/2}V''(z)
\end{aligned}
$$

Thus the IVP becomes

$$V''(z) + \frac{1}{2}(zV(z))' = 0.$$

Integration

Integrating once yields

$$V'(z) + \frac{z}{2}V(z) = C$$

where C is a constant. Integrate once again with the aid of the integrating factor $e^{-z^2/4}$ to obtain

$$V(z) = Ce^{-z^2/4}\int e^{-s^2/4}\,ds + De^{-z^2/4}.$$

Choose $C = 0$, then

$$u(x,\tau) = \frac{D}{\sqrt{\tau}}e^{-x^2/(4\tau)}.$$

Normalization

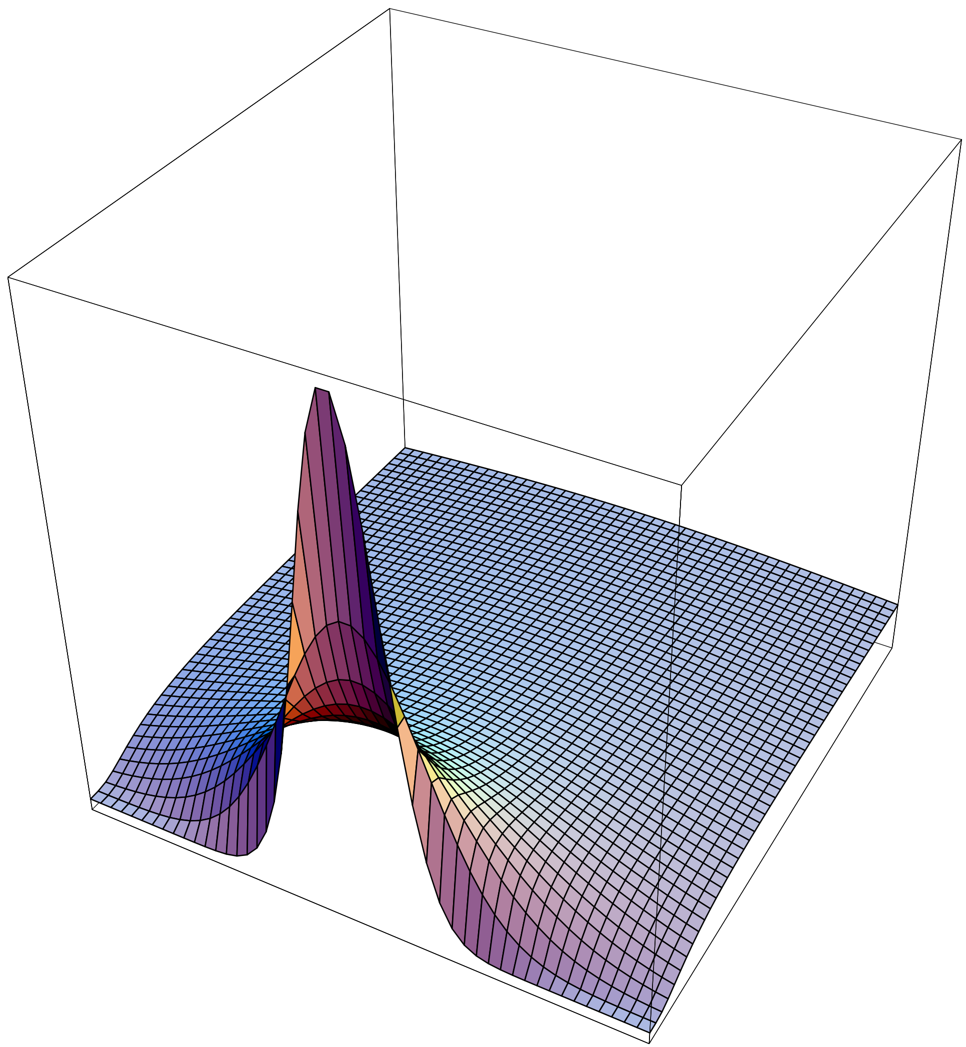
Normalize the solution using the result that

$$\int_{-\infty}^{\infty} e^{-x^2/(4\tau)}\,dx = 2\sqrt{\pi\tau} \qquad \text{hence,}$$

$$u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}} e^{-x^2/(4\tau)}.$$

Note: Think of an infinitely long insulated rod initially containing one unit of heat concentrated at the origin.

Visualization of Fundamental Solution



Superposition of Solutions

Now consider the heat equation with more general initial data:

$$
\begin{aligned}
u_\tau &= u_{xx} && \text{for } -\infty < x < \infty,\ \tau > 0 \\
u(x,0) &= u_0(x) && \text{for } -\infty < x < \infty \\
\lim_{|x|\to\infty} u(x,\tau) &= 0 && \text{for } \tau > 0.
\end{aligned}
$$

The Dirac delta function has the property,

$$
u_0(x) = \int_{-\infty}^{\infty} u_0(s)\delta(s-x)\, ds.
$$

Solution for General ICs

The heat equation is linear so superposition of solutions holds. Note that

$$u_0(s)\frac{1}{2\sqrt{\pi\tau}}e^{-(s-x)^2/(4\tau)}$$

solves the heat equation with initial condition $u_0(s)\delta(s-x)$. Thus the solution to the heat equation,

$$u(x,\tau)=\frac{1}{2\sqrt{\pi\tau}}\int_{-\infty}^{\infty}u_0(s)e^{-(s-x)^2/(4\tau)}\,ds,$$

satisfies the initial condition

$$u(x,0)=\int_{-\infty}^{\infty}u_0(s)\delta(s-x)\,ds=u_0(x).$$

Let $z = (s - x)/\sqrt{2\tau}$ and then

$$\begin{aligned} u(x,\tau) &= \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} u_0(x + z\sqrt{2\tau}) e^{-z^2/2} \sqrt{2\tau}\, dz \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Big(e^{(k+1)(x+z\sqrt{2\tau})/2} \\ &\qquad - e^{(k-1)(x+z\sqrt{2\tau})/2} \Big)^{+} e^{-z^2/2}\, dz \\ &= \frac{1}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} \Big(e^{(k+1)(x+z\sqrt{2\tau})/2} \\ &\qquad - e^{(k-1)(x+z\sqrt{2\tau})/2} \Big) e^{-z^2/2}\, dz \\ &= I_1 - I_2 \end{aligned}$$

I_1 Derivation

$$
\begin{aligned}
I_1 &= \frac{1}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{(k+1)(x+z\sqrt{2\tau})/2} e^{-z^2/2}\, dz \\
&= \frac{e^{(k+1)x/2}}{\sqrt{2\tau}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{-(z^2 - z(k+1)\sqrt{2\tau})/2}\, dz \\
&= \frac{e^{(k+1)x/2}}{\sqrt{2\tau}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{(k+1)^2\tau/4} e^{-(z^2-(k+1)\sqrt{\tau/2})^2/2}\, dz \\
&= \frac{e^{(k+1)x/2+(k+1)^2\tau/4}}{\sqrt{2\tau}} \int_{-x/\sqrt{2\tau}-(k+1)\sqrt{\tau/2}}^{\infty} e^{-y^2/2}\, dy \\
&= e^{(k+1)x/2+(k+1)^2\tau/4} \phi(w)
\end{aligned}
$$

I_2 Derivation

Where

$$\phi(z) = \frac{1}{\sqrt{2\tau}} \int_{-\infty}^{z} e^{-\eta^2/2}\, d\eta$$

and $$w = \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k+1)\sqrt{2\tau}$$

Similarly we can derive

$$I_2 = e^{(k-1)x/2+(k-1)^2\tau/4}\phi(w - \sqrt{2\tau}).$$

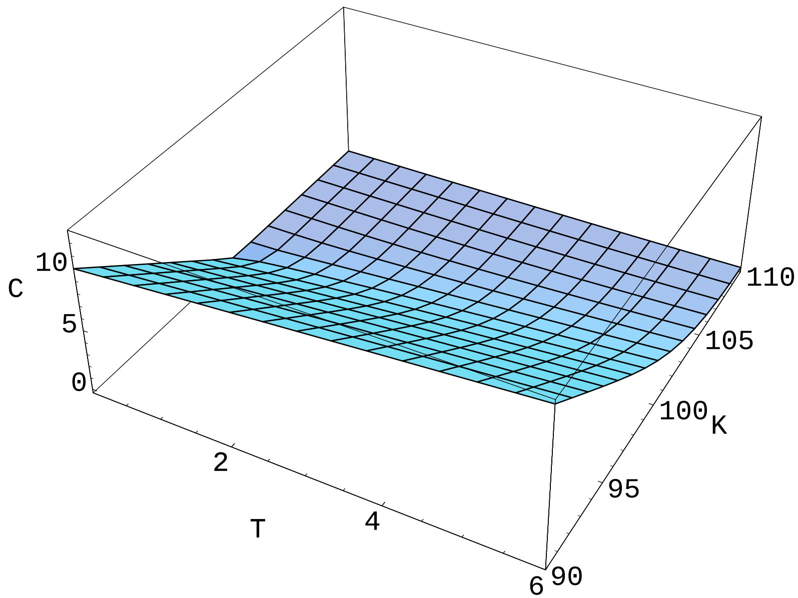
Change of Variables Redux

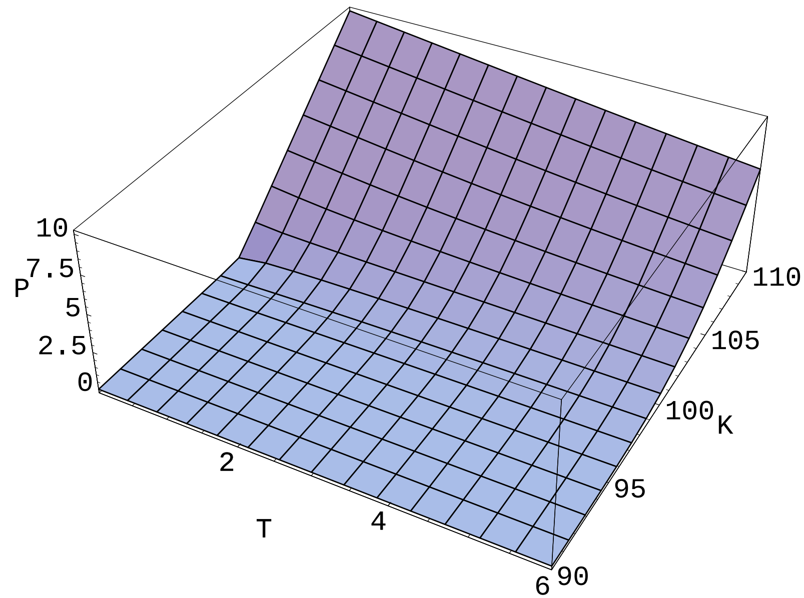
Now we must undo all the changes of variables.

$$
\begin{aligned}
u(x,\tau) &= e^{(k+1)x/2+(k+1)^2\tau/4}\phi(w) \\
&\qquad -e^{(k-1)x/2+(k-1)^2\tau/4}\phi(w-\sqrt{2\tau}) \\
v(x,\tau) &= e^{-(k-1)x/2-(k+1)^2\tau/4}u(x,\tau) \\
&= e^{x}\phi(w) - e^{-k\tau}\phi(w-\sqrt{2\tau}) \\
v(S,t) &= \frac{S}{K}\phi(w) - e^{-r(T-t)}\phi(w-\sigma\sqrt{T-t}) \\
C(S,t) &= Kv(S,t) \\
&= S\phi(w) - Ke^{-r(T-t)}\phi(w-\sigma\sqrt{T-t})
\end{aligned}
$$

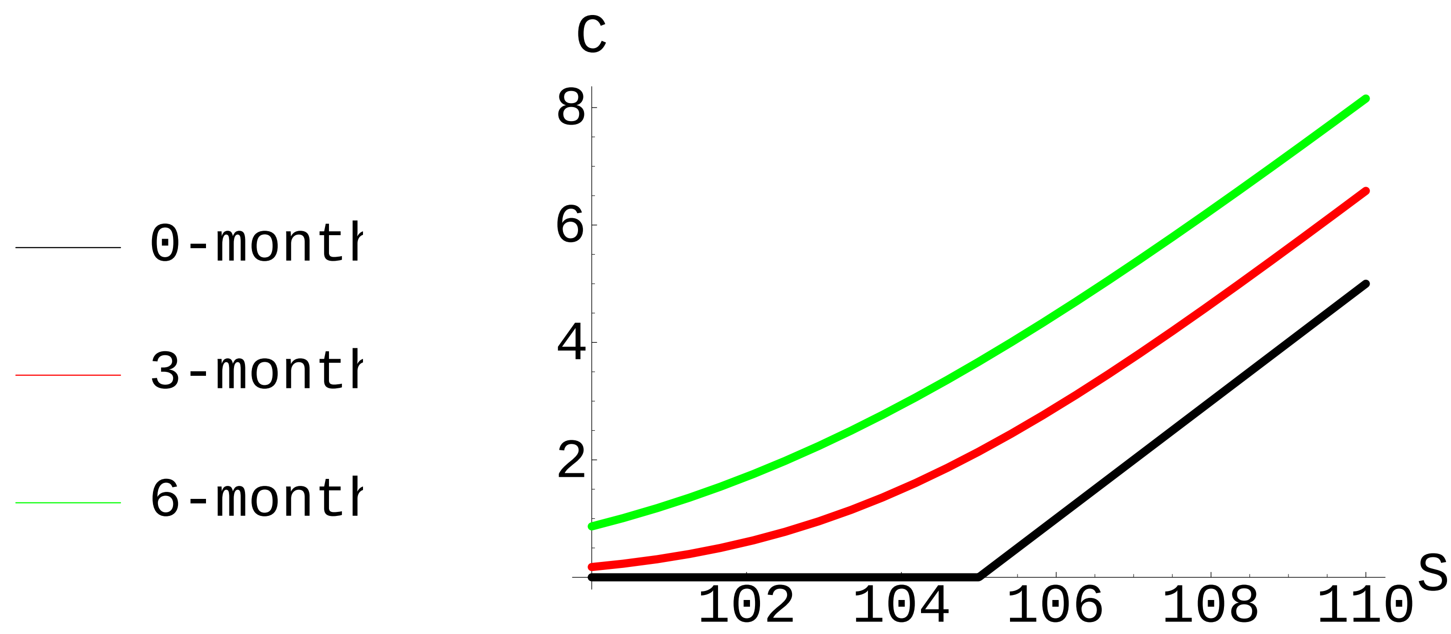
where $w = \dfrac{1}{\sigma\sqrt{T-t}}\left[(r+\dfrac{\sigma^2}{2})(T-t)+\ln(S/K)\right]$.

Sensitivity of Option Prices





Time Dependency of Option Prices



Are these prices real . . .

. . . or do arbitrage opportunities exist?

- Black and Scholes (1972) showed option prices can deviate from those given in their formula, but the profit was eliminated when transaction costs were considered.
- Galai (1977) confirmed that 1% transaction costs eliminate excess profit.
- Bhattacharya (1983) also confirmed.
- MacBeth and Merville (1979) observed systematic deviations of prices for long time to expiration and options way in- or way out-of-the-money.

American Options

Recall:

- A European Option, if exercised at all, can only be exercised at time $t = T$.
- An American Option, if exercised at all, can be exercised for any $0 \leq t \leq T$.

Consequences: In an arbitrage-free setting

1. $C_a \geq C_e$ and $P_a \geq P_e$
2. $C_a \geq C_e \geq S - Ke^{-r(T-t)}$

(If $C_e < S - Ke^{-r(T-t)}$ equivalent to $K < (S - C_e)e^{r(T-t)}$, the profit from shorting the security, purchasing the call, and investing the balance from the exercise time t until expiry T.)

Early Exercise

Claim: For a non-dividend paying security, early exercise of an American call is never advantageous.
By the previous result

$$C_a \geq S - Ke^{-r(T-t)} > S - K,$$

if the option is exercised at $t < T$. Thus $C_a + K > S$, *i.e.* the American call and a cash amount K is worth more than the stock just purchased.

Consequently $C_a = C_e$ for non-dividend paying securities.

American Put-Call “Parity”

For American options an inequality is satisfied,

$$S - K \leq C_a - P_a \leq S - Ke^{-r(T-t)}.$$

If $S - K > C_a - P_a$, short S, sell P_a, buy C_a, invest the proceeds at interest rate r. If the owner of the put exercises at time t, the net gain is

$$(S + P_a - C_a)e^{rt} - K > (S + P_a - C_a - K)e^{rt} > 0.$$

Since $C_a = C_e$ for a non-dividend paying security and $P_a \geq P_e$ then the other inequality is a consequence of the European put-call parity formula.

Closing Thoughts

1. Dividend paying securities
2. Pricing of American options
3. Time-varying μ, σ, r
4. Development of a calculus-free course