# 1 Moving Average Time Series Model

Suppose $\{Z_t\}_{t=1}^n$ is a sample of a scalar time series (in other words for a fixed $t$, $Z_t$ is a scalar as opposed to vector quantity). We would like to model the time series using a **moving average** time series model. When $q$ is a positive integer, a so-called "MA($q$)" model is one of the form

$$Z_t = \theta_0 + a_t - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q}$$

for $t = 1, \ldots, n$ where $\{a_t\}_{t=1-q}^n$ is a scalar "white noise" series modeled for each $t$ by a normally distributed random variable with zero mean and variance $\sigma_a^2$.

Taking the expected value of both sides of the previous equation yields

$$\mathrm{E}\left[Z_t\right] = \theta_0$$

thus the sample mean of the series $\overline{Z_t}$, provides an estimate of $\theta_0$. The remaining parameters of the model are $\theta_1, \ldots, \theta_q$, the preliminary values of the "shock" series $a_{1-q}, \ldots, a_0$ and $\sigma_a$.

# 2 Matrix/Vector Formulation

Our goal is to numerically estimate the parameter values of the MA($q$) model for a given time series $\{Z_t\}_{t=1}^n$. We may write the time series in "system" format as

$$\begin{aligned}
Z_1 &= \theta_0 + a_1 - \theta_1 a_0 - \cdots - \theta_q a_{1-q} \\
Z_2 &= \theta_0 + a_2 - \theta_1 a_1 - \cdots - \theta_q a_{2-q} \\
&\vdots \\
Z_n &= \theta_0 + a_n - \theta_1 a_{n-1} - \cdots - \theta_q a_{n-q}.
\end{aligned}$$

For the sake of convenience we will define $z_t \equiv Z_t - \theta_0$ and rewrite these equations once again.

$$\begin{aligned}
z_1 &= a_1 - \theta_1 a_0 - \cdots - \theta_q a_{1-q} \\
z_2 &= a_2 - \theta_1 a_1 - \cdots - \theta_q a_{2-q} \\
&\vdots \\
z_n &= a_n - \theta_1 a_{n-1} - \cdots - \theta_q a_{n-q}
\end{aligned}$$

Adopting the notation of [Hillmer and Tiao (1979)] we can write the above system of equations in matrix/vector form.

$$\mathbf{z} = D(\boldsymbol{\theta})\mathbf{a} - \Delta C(\boldsymbol{\theta})\mathbf{a}_*. \tag{1}$$

The vector $\mathbf{a}_* = \langle a_{1-q}, \ldots, a_0 \rangle$ is sometimes referred to as the vector of preliminary values of the shocks. The remaining vector is $\mathbf{a} = \langle a_1, \ldots, a_n \rangle$. Naturally $\mathbf{z} = \langle z_1, \ldots, z_n \rangle$. Matrix $D(\boldsymbol{\theta})$ is an $n \times n$ lower triangular matrix whose entries depend on $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_q)$.

$$D(\boldsymbol{\theta}) = \begin{bmatrix}
1 & & & & 0 \\
-\theta_1 & 1 & & & \\
\vdots & & \ddots & & \\
-\theta_q & & \ddots & & \\
 & \ddots & & & \\
0 & & -\theta_q & \cdots & -\theta_1 & 1
\end{bmatrix}$$

$C(\boldsymbol{\theta})$ is a $q \times q$ upper triangular matrix with the form

$$C(\boldsymbol{\theta}) = \begin{bmatrix} \theta_q & \theta_{q-1} & \cdots & \theta_1 \\ & \ddots & \ddots & \vdots \\ & & & \theta_{q-1} \\ 0 & & & \theta_q \end{bmatrix}.$$

Finally, $\Delta$ is an $n \times q$ block matrix whose topmost block is the $q \times q$ identity matrix. The remaining entries of the matrix are zero.

# 3 Parameter Estimation

Using Eq. (1) we may define the function

$$f(\boldsymbol{\theta}, \mathbf{a}_*, \sigma_a) = (\mathbf{z} - D(\boldsymbol{\theta})\mathbf{a} + \Delta C(\boldsymbol{\theta})\mathbf{a}_*)^T (\mathbf{z} - D(\boldsymbol{\theta})\mathbf{a} + \Delta C(\boldsymbol{\theta})\mathbf{a}_*).$$

There are $2q + 1$ parameters ($\theta_1, \ldots, \theta_q$ and $a_{1-q}, \ldots, a_0$ and $\sigma_a$) to be estimated. Many authors describe the estimation process in terms of a Newton-Raphson iterative process. The steps outlined in [Box and Jenkins (1970)]–[Tsay (2005)] seem to be as follows.

1. Given an initial estimate of $\boldsymbol{\theta}$ and $\sigma_a$, find the vector $\hat{\mathbf{a}}_*$ which minimizes $f(\boldsymbol{\theta}, \mathbf{a}_*, \sigma_a)$.

   The first time through this step there is no prior estimate of $\boldsymbol{\theta}$ or $\sigma_a$, thus arbitrarily the initial estimate of $\sigma_a$ will be chosen to be 1 and for $i = 1, \ldots, q$, $\theta_i$ will be selected randomly from a uniform distribution of real numbers in the interval $(-1, 1)$. Once the value of $\sigma_a$ is selected the vector of shocks $\mathbf{a}$ can be realized as a vector whose entries are random real numbers chosen from a normal distribution with mean zero and standard deviation $\sigma_a$.

2. Once $\hat{\mathbf{a}}_*$ is calculated it is used to evaluate $\hat{\mathbf{a}}$ using the formula below.

   $$\hat{\mathbf{a}} = D^{-1}(\boldsymbol{\theta}) \left(\Delta C(\boldsymbol{\theta})\hat{\mathbf{a}}_* + \mathbf{z}\right)$$

   [Hillmer and Tiao (1979)] contains a formula for the direct evaluation of $D^{-1}(\boldsymbol{\theta})$, thus no matrix inversion is required.

3. Now that $\hat{\mathbf{a}}_*$ and $\hat{\mathbf{a}}$ have been estimated, a new estimate of $\sigma_a$ may be calculated. For a random variable $X$, the variance of $X$ is $\mathrm{Var}(X) = \mathrm{E}\left[X^2\right] - \left(\mathrm{E}\left[X\right]\right)^2$, and thus

   $$\widehat{\sigma_a}^2 = \frac{1}{n} \sum_{t=1-q}^{n} \hat{a}_t^2 = \frac{1}{n} \left(\hat{\mathbf{a}}_*, \hat{\mathbf{a}}\right)^T \left(\hat{\mathbf{a}}_*, \hat{\mathbf{a}}\right).$$

4. Now that estimates of $\sigma_a$ and $\mathbf{a}_*$ are available and the vector of shocks $\mathbf{a}$ has been calculated, they are substituted into function $f$ and a vector $\hat{\boldsymbol{\theta}}$ is found which minimizes $f(\boldsymbol{\theta}, \mathbf{a}_*, \sigma_a)$.

5. If necessary the estimates of $\boldsymbol{\theta}$, $\mathbf{a}_*$, and $\sigma_a$ may be improved by returning to step 1 with the current estimates and iterating the process.

## References

[Box and Jenkins (1970)] Box, George E.P. and Jenkins, Gwilym M. (1970), *Time Series Analysis: forecasting and control*, Holden-Day, San Francisco, California, USA.

[Hillmer and Tiao (1979)] Hillmer, Steven C. and Tiao, George C. (1979), Likelihood Function of Stationary Multiple Autoregressive Moving Average Models, *Journal of the American Statistical Association*, **74**, 367, pp. 652–660.

[Tsay (2005)] Tsay, Ruey S. (2005), *Analysis of Financial Time Series*, 2nd edition, John Wiley & Sons, Inc., Hoboken, New Jersey, USA.