Initial Value Problem for the European Call

\[ rF = F_t + rSF_S + \frac{1}{2} \sigma^2 S^2 F_{SS} \quad \text{for} \quad (S, t) \in [0, \infty) \times [0, T], \]

\[ F(S, T) = (S(T) - K)^+ \quad \text{for} \quad S > 0, \]

\[ F(0, t) = 0 \quad \text{for} \quad 0 \leq t < T, \]

\[ F(S, t) = S - Ke^{-r(T-t)} \quad \text{as} \quad S \to \infty. \]

We will solve this system of equations using Fourier Transforms.
If \( f : \mathbb{R} \rightarrow \mathbb{R} \) then the **Fourier Transform** of \( f \) is

\[
\mathcal{F}\{f(x)\} = \hat{f}(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} \, dx,
\]

where \( i = \sqrt{-1} \) and \( \omega \) is a parameter. The Fourier Transform exists only if the improper integral converges.
Definition

If $f : \mathbb{R} \to \mathbb{R}$ then the **Fourier Transform** of $f$ is

$$
\mathcal{F}\{f(x)\} = \hat{f}(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} \, dx,
$$

where $i = \sqrt{-1}$ and $\omega$ is a parameter. The Fourier Transform exists only if the improper integral converges.

The Fourier Transform of $f$ will exist when

- $f$ and $f'$ are piecewise continuous on every interval of the form $[-M, M]$ for arbitrary $M > 0$, and
- $\int_{-\infty}^{\infty} |f(x)| \, dx$ converges.
When working with complex-valued exponentials, the Euler Identity may be useful:

\[ e^{i\theta} = \cos \theta + i \sin \theta. \]

Example

Find the Fourier Transform of the piecewise-defined function

\[
f(x) = \begin{cases} 
1/2 & \text{if } |x| \leq 1, \\
0 & \text{otherwise.}
\end{cases}
\]
\[ \hat{f}(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} \, dx \]

\[ = \int_{-1}^{1} \frac{1}{2} e^{-i\omega x} \, dx \]

\[ = \frac{-1}{2i\omega} e^{-i\omega x} \Bigg|_{-1}^{1} \]

\[ = \frac{-1}{2i\omega} (e^{-i\omega} - e^{i\omega}) \]

\[ = \frac{1}{\omega} \left( \frac{e^{i\omega} - e^{-i\omega}}{2i} \right) \]

\[ = \frac{1}{\omega} \left( \frac{\cos \omega + i \sin \omega - \cos \omega + i \sin \omega}{2i} \right) \]

\[ = \frac{\sin \omega}{\omega} \]
Example

Suppose the Fourier Transform of $f$ exists and that $f'$ exists, find $\mathcal{F}\{f'(x)\}$. 
Applying integration by parts with

\[ u = e^{-i\omega x} \quad v = f(x) \]
\[ du = -i\omega e^{-i\omega x} \, dx \quad dv = f'(x) \, dx \]

\[ \mathcal{F} \{ f'(x) \} = \int_{-\infty}^{\infty} f'(x) e^{-i\omega x} \, dx \]

\[ = f(x)e^{-i\omega x} \bigg|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)(-i\omega) e^{-i\omega x} \, dx \]

\[ = i\omega \int_{-\infty}^{\infty} f(x)e^{-i\omega x} \, dx \]

\[ = i\omega \hat{f}(\omega). \]
Fourier Transforms and Derivatives

Theorem

If \( f(x) \), \( f'(x) \), \( \ldots \), \( f^{(n-1)}(x) \) are all Fourier transformable and if \( f^{(n)}(x) \) exists (where \( n \in \mathbb{N} \)) then \( \mathcal{F}\{ f^{(n)}(x) \} = (i\omega)^n \hat{f}(\omega) \).
The previous example demonstrates the result is true for $n = 1$. Suppose the result is true for $n = k \geq 1$. By definition

$$\mathcal{F}\left\{f^{(k+1)}(x)\right\} = \int_{-\infty}^{\infty} f^{(k+1)}(x) e^{-i\omega x} \, dx$$

$$= f^{(k)}(x) e^{-i\omega x}\bigg|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f^{(k)}(x)(-i\omega) e^{-i\omega x} \, dx$$

$$= (i\omega) \int_{-\infty}^{\infty} f^{(k)}(x) e^{-i\omega x} \, dx$$

$$= (i\omega)(i\omega)^k \hat{f}(\omega)$$

$$= (i\omega)^{k+1} \hat{f}(\omega).$$

The result follows by induction on $k$. 
The **Fourier Convolution** of two functions $f$ and $g$ is

$$(f \ast g)(x) = \int_{-\infty}^{\infty} f(x - z)g(z) \, dz,$$

provided the improper integral converges.
Fourier Convolution

Definition

The **Fourier Convolution** of two functions $f$ and $g$ is

$$(f \ast g)(x) = \int_{-\infty}^{\infty} f(x - z)g(z) \, dz,$$

provided the improper integral converges.

Theorem

$\mathcal{F}\{ (f \ast g)(x) \} = \hat{f}(\omega)\hat{g}(\omega)$, in other words the Fourier Transform of the Fourier Convolution of $f$ and $g$ is the product of the Fourier Transforms of $f$ and $g$. 

Solving the Black-Scholes Equation
\[ \mathcal{F}\{(f \ast g)(x)\} = \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} f(x - z)g(z) \, dz \right] e^{-i\omega x} \, dx \]

\[ = \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} f(x - z)g(z)e^{-i\omega x} \, dz \right] \, dx \]

\[ = \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} f(x - z)g(z)e^{-i\omega x} \, dx \right] \, dz \]

\[ = \int_{-\infty}^{\infty} g(z) \left[ \int_{-\infty}^{\infty} f(x - z)e^{-i\omega x} \, dx \right] \, dz \]
Proof (2 of 2)

So far,

\[
\mathcal{F}\{(f \ast g)(x)\} = \int_{-\infty}^{\infty} g(z) \left[ \int_{-\infty}^{\infty} f(x - z) e^{-i\omega x} \, dx \right] \, dz \\
= \int_{-\infty}^{\infty} g(z) \left[ \int_{-\infty}^{\infty} f(u) e^{-i\omega (u+z)} \, du \right] \, dz \\
= \int_{-\infty}^{\infty} g(z) e^{-i\omega z} \left[ \int_{-\infty}^{\infty} f(u) e^{-i\omega u} \, du \right] \, dz \\
= \hat{f}(\omega) \int_{-\infty}^{\infty} g(z) e^{-i\omega z} \, dz \\
= \hat{f}(\omega) \hat{g}(\omega)
\]
The inverse Fourier Transform of $\hat{f}(\omega)$ given by

$$\mathcal{F}^{-1}\{\hat{f}(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} \, d\omega.$$
Example

Find the inverse Fourier Transform of $e^{-|\omega|}$. 
\[
\mathcal{F}^{-1}\left\{e^{-|\omega|}\right\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-|\omega|} e^{i\omega x} d\omega
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{0} e^{(1+ix)\omega} d\omega + \frac{1}{2\pi} \int_{0}^{\infty} e^{(-1+ix)\omega} d\omega
\]

\[
= \frac{1}{2\pi} \frac{1}{1 + ix} e^{(1+ix)\omega}\bigg|_{-\infty}^{0} + \frac{1}{2\pi} \frac{1}{-1 + ix} e^{(-1+ix)\omega}\bigg|_{0}^{\infty}
\]

\[
= \frac{1}{2\pi(1 + ix)} + \frac{1}{2\pi(1 - ix)}
\]

\[
= \frac{1}{\pi(1 + x^2)}
\]
We will use the Fourier Transform and its inverse to solve the Black-Scholes PDE once we have performed a suitable change of variables on the PDE.

Let

\[ x = \ln \frac{S}{K} \]
\[ \tau = \frac{\sigma^2}{2}(T - t) \]

\[ v(x, \tau) = \frac{1}{K} F(S, t) \]

and calculate \( F_t, F_S, \) and \( F_{SS} \).
Change of Variables

\[ F_t = -\frac{K\sigma^2}{2} v_T \]
\[ F_S = e^{-x} v_x \]
\[ F_{SS} = \frac{e^{-2x}}{K} (v_{xx} - v_x) \]
Change of Variables

\[ F_t = -\frac{K\sigma^2}{2} v_T \]
\[ F_S = e^{-x} v_x \]
\[ F_{SS} = \frac{e^{-2x}}{K} (v_{xx} - v_x) \]

Substitute into the Black-Scholes PDE:

\[ rF = F_t + rSF_S + \frac{1}{2} \sigma^2 S^2 F_{SS} \]
Change of Variables

\[ F_t = -\frac{K \sigma^2}{2} v_\tau \]
\[ F_S = e^{-x} v_x \]
\[ F_{SS} = \frac{e^{-2x}}{K} (v_{xx} - v_x) \]

Substitute into the Black-Scholes PDE:

\[ rF = F_t + rSF_S + \frac{1}{2} \sigma^2 S^2 F_{SS} \]
\[ v_\tau = v_{xx} + (k - 1) v_x - kv \]

where \( k = \frac{2r}{\sigma^2} \).
The final condition

\[ F(S, T) = (S(T) - K)^+ \]
\[ Kv(x, 0) = (Ke^x - K)^+ \]
\[ v(x, 0) = (e^x - 1)^+ \]

becomes an initial condition.
The final condition

\[ F(S, T) = (S(T) - K)^+ \]
\[ Kv(x, 0) = (Ke^x - K)^+ \]
\[ v(x, 0) = (e^x - 1)^+ \]

becomes an initial condition.

The boundary condition

\[ F(0, t) = \lim_{S \to 0^+} F(S, t) \]
\[ 0 = \lim_{x \to -\infty} Kv(x, \tau) \]
\[ 0 = \lim_{x \to -\infty} v(x, \tau) \].
The boundary condition

\[
\lim_{S \to \infty} F(S, t) = S - Ke^{-r(T-t)} \\
\lim_{x \to \infty} K\nu(x, \tau) = Ke^x - Ke^{-r(T-[T-2\tau/\sigma^2])} \\
\lim_{x \to \infty} \nu(x, \tau) = e^x - e^{-k\tau}.
\]
The boundary condition

\[
\lim_{S \to \infty} F(S, t) = S - Ke^{-r(T-t)} \\
\lim_{x \to \infty} Kv(x, \tau) = Ke^{x} - Ke^{-r(T-[T-2\tau/\sigma^2])} \\
\lim_{x \to \infty} v(x, \tau) = e^{x} - e^{-k\tau}.
\]

The initial value problem in the new variables is

\[
v_{\tau} = v_{xx} + (k - 1)v_{x} - kv \quad \text{for } x \in (-\infty, \infty), \ \tau \in (0, \frac{T\sigma^2}{2}) \\
v(x, 0) = (e^{x} - 1)^{+} \quad \text{for } x \in (-\infty, \infty) \\
v(x, \tau) \to 0 \quad \text{as } x \to -\infty \text{ and} \\
v(x, \tau) \to e^{x} - e^{-k\tau} \quad \text{as } x \to \infty, \ \tau \in (0, \frac{T\sigma^2}{2})
\]
Suppose $\alpha$ and $\beta$ are constants and

$$v(x, \tau) = e^{\alpha x + \beta \tau} u(x, \tau).$$

Find $v_x$, $v_{xx}$, and $v_\tau$ in terms of $u_x$, $u_{xx}$, and $u_\tau$. 
Suppose \( \alpha \) and \( \beta \) are constants and

\[
v(x, \tau) = e^{\alpha x + \beta \tau} u(x, \tau).
\]

Find \( v_x \), \( v_{xx} \), and \( v_\tau \) in terms of \( u_x \), \( u_{xx} \), and \( u_\tau \).

\[
v_x = e^{\alpha x + \beta \tau} \left( \alpha u(x, \tau) + u_x \right)
\]

\[
v_{xx} = e^{\alpha x + \beta \tau} \left( \alpha^2 u(x, \tau) + 2\alpha u_x + u_{xx} \right)
\]

\[
v_\tau = e^{\alpha x + \beta \tau} \left( \beta u(x, \tau) + u_\tau \right)
\]
Substituting into the PDE

\[ v_\tau = v_{xx} + (k - 1)v_x - kv \]
\[ u_\tau = (\alpha^2 + (k - 1)\alpha - k - \beta)u + (2\alpha + k - 1)u_x + u_{xx} \]
Substituting into the PDE

\[ v_\tau = v_{xx} + (k - 1)v_x - kv \]
\[ u_\tau = (\alpha^2 + (k - 1)\alpha - k - \beta)u + (2\alpha + k - 1)u_x + u_{xx} \]

If we choose \( \alpha \) and \( \beta \) so that

\[ 0 = \alpha^2 + (k - 1)\alpha - k - \beta \]
\[ 0 = 2\alpha + k - 1 \]

then we have the PDE

\[ u_\tau = u_{xx} \]

which is known as the **Heat Equation**.
Let $\alpha = \frac{1 - k}{2}$ and $\beta = -\frac{(k + 1)^2}{4}$, then the initial condition becomes:

$$n(x, 0) = (e^x - 1)^+$$
$$u(x, 0) = (e^{(k+1)x/2} - e^{(k-1)x/2})^+. $$
Side Conditions

Let \( \alpha = \frac{1-k}{2} \) and \( \beta = -\frac{(k+1)^2}{4} \), then the initial condition becomes:

\[
\begin{align*}
v(x, 0) &= (e^x - 1)^+ \\
u(x, 0) &= (e^{(k+1)x/2} - e^{(k-1)x/2})^+.
\end{align*}
\]

The boundary conditions become

\[
\begin{align*}
\lim_{x \to -\infty} v(x, \tau) &= 0 \\
\lim_{x \to -\infty} u(x, \tau) &= 0
\end{align*}
\]

and

\[
\begin{align*}
\lim_{x \to \infty} v(x, \tau) &= e^x - e^{-k\tau} \\
\lim_{x \to \infty} u(x, \tau) &= e^{\frac{(k+1)}{2}[x+(k+1)\tau/2]} - e^{\frac{(k-1)}{2}[x+(k-1)\tau/2]}.
\end{align*}
\]
\begin{align*}
  u_\tau &= u_{xx} \quad \text{for } x \in (-\infty, \infty) \text{ and } \tau \in (0, T\sigma^2/2) \\
  u(x, 0) &= (e^{(k+1)x/2} - e^{(k-1)x/2})^+ \quad \text{for } x \in (-\infty, \infty) \\
  u(x, \tau) &\to 0 \quad \text{as } x \to -\infty \text{ for } \tau \in (0, T\sigma^2/2) \\
  u(x, \tau) &\to e^{\frac{k+1}{2}[x+(k+1)\tau/2]} - e^{\frac{k-1}{2}[x+(k-1)\tau/2]}
\end{align*}
Solving the Heat Equation

\[ u_\tau = u_{xx} \]
\[ \mathcal{F}\{u_\tau\} = \mathcal{F}\{u_{xx}\} \]
\[ \frac{d\hat{u}}{d\tau} = -\omega^2 \hat{u} \]
\[ \hat{u}(\omega, \tau) = De^{-\omega^2\tau} \]

where \( D = \hat{f}(\omega) \) is the Fourier Transform of the initial condition.
Recall the Fourier Convolution and the Fourier Transform of the Fourier Convolution.

\[
\mathcal{F}^{-1}\{\hat{u}(\omega, \tau)\} = \mathcal{F}^{-1}\{\hat{f}(\omega) e^{-\omega^2 \tau}\}
\]

\[
u(x, \tau) = (e^{(k+1)x/2} - e^{(k-1)x/2}) \ast \frac{1}{2\sqrt{\pi\tau}} e^{-x^2/(4\tau)}
\]

\[
= \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} (e^{(k+1)z/2} - e^{(k-1)z/2}) e^{-\frac{(x-z)^2}{4\tau}} dz
\]
Make the substitutions:

\[ z = x + \sqrt{2\tau} y \]
\[ dz = \sqrt{2\tau} \, dy \]

then

\[
u(x, \tau) = \frac{1}{2 \sqrt{\pi \tau}} \int_{-\infty}^{\infty} \left( e^{(k+1)\frac{z}{2}} - e^{(k-1)\frac{z}{2}} \right) + e^{-\frac{(x-z)^2}{4\tau}} \, dz
\]

\[
= \frac{e^{(k+1)x/2} e^{(k+1)^2\tau/4}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left( y - \frac{1}{2} (k+1) \sqrt{2\tau} \right)^2/2} \, dy
\]

\[
- \frac{e^{(k-1)x/2} e^{(k-1)^2\tau/4}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left( y - \frac{1}{2} (k-1) \sqrt{2\tau} \right)^2/2} \, dy
\]
Now make the substitutions \( w = y - \frac{1}{2}(k + 1)\sqrt{2\tau} \) in the first integral and \( w' = y - \frac{1}{2}(k - 1)\sqrt{2\tau} \) in the second.

\[
\begin{align*}
    u(x, \tau) &= \frac{e^{(k+1)x/2}e^{(k+1)^2\tau/4}}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{-(y - \frac{1}{2}(k+1)\sqrt{2\tau})^2/2} \, dy \\
    &\quad - \frac{e^{(k-1)x/2}e^{(k-1)^2\tau/4}}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{-(y - \frac{1}{2}(k-1)\sqrt{2\tau})^2/2} \, dy \\
    &= e^{(k+1)x/2+(k+1)^2\tau/4} \phi \left( \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k + 1)\sqrt{2\tau} \right) \\
    &\quad - e^{(k-1)x/2+(k-1)^2\tau/4} \phi \left( \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k - 1)\sqrt{2\tau} \right)
\end{align*}
\]
Note that
\[ e^{(k+1)\frac{x}{2}+(k+1)^2\frac{\tau}{4}} e^{-(k-1)\frac{x}{2}-(k+1)^2\frac{\tau}{4}} = e^x \]
\[ e^{(k-1)\frac{x}{2}+(k-1)^2\frac{\tau}{4}} e^{-(k-1)\frac{x}{2}-(k+1)^2\frac{\tau}{4}} = e^{-k\tau} \]

and therefore
\[ v(x, \tau) = e^{-(k-1)x/2-(k+1)^2\tau/4} u(x, \tau) \]
\[ = e^x \phi \left( \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k+1)\sqrt{2\tau} \right) \]
\[ - e^{-k\tau} \phi \left( \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k-1)\sqrt{2\tau} \right). \]
Recall that

\[ x = \ln \frac{S}{K} \]
\[ \tau = \frac{\sigma^2}{2}(T - t) \]
\[ k = \frac{2r}{\sigma^2} \]

and thus

\[ \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k + 1)\sqrt{2\tau} = \frac{\ln(S/K) + (r + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}} = w \]
\[ \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k - 1)\sqrt{2\tau} = w - \sigma\sqrt{T - t}. \]
\[ v(x, \tau) = \frac{S}{K} \phi(w) - e^{-r(T-t)} \phi(w - \sigma \sqrt{T-t}) \]

\[ F(S, t) = S \phi(w) - Ke^{-r(T-t)} \phi(w - \sigma \sqrt{T-t}) \]
Undoing the Change of Variables (5 of 5)

\[ v(x, \tau) = \frac{S}{K} \phi(w) - e^{-r(T-t)} \phi(w - \sigma \sqrt{T-t}) \]
\[ F(S, t) = S\phi(w) - Ke^{-r(T-t)} \phi(w - \sigma \sqrt{T-t}) \]

Finally we have the formulas for the European call and put.

\[ C(S, t) = S\phi(w) - Ke^{-r(T-t)} \phi(w - \sigma \sqrt{T-t}) \]
\[ P(S, t) = Ke^{-r(T-t)} \phi(\sigma \sqrt{T-t} - w) - S\phi(-w) \]
Plotting the Call Price

\[ \text{K} - \text{LParen}1 \text{K}\text{RParen}1 \text{S}\text{T} \text{t} \]

J. Robert Buchanan

Solving the Black-Scholes Equation
Plotting the Put Price

Solving the Black-Scholes Equation
Example

Suppose the current price of a security is $62 per share. The continuously compounded interest rate is 10% per year. The volatility of the price of the security is $\sigma = 20\%$ per year. Find the cost of a five-month European call option with a strike price of $60$ per share.
Summary:

\[ T = \frac{5}{12}, \quad t = 0, \quad r = 0.10, \]
\[ \sigma = 0.20, \quad S = 62, \quad \text{and} \quad K = 60. \]
Summary:

\[ T = \frac{5}{12}, \quad t = 0, \quad r = 0.10, \quad \sigma = 0.20, \quad S = 62, \quad \text{and} \quad K = 60. \]

\[ w = \ln \left( \frac{S}{K} \right) + \frac{(r + \sigma^2/2)(T - t)}{\sigma \sqrt{T - t}} \]

\[ C = S\phi(w) - Ke^{-r(T-t)}\phi(w - \sigma \sqrt{T - t}) \]
Summary:

\[ T = \frac{5}{12}, \quad t = 0, \quad r = 0.10, \]
\[ \sigma = 0.20, \quad S = 62, \quad \text{and} \quad K = 60. \]

\[
w = \ln\left(\frac{S}{K}\right) + \frac{(r + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}} \approx 0.641287
\]

\[
C = S\phi(w) - Ke^{-r(T-t)}\phi(w - \sigma\sqrt{T - t}) \approx $5.80
\]
Example

Suppose the current price of a security is $97 per share. The continuously compounded interest rate is 8% per year. The volatility of the price of the security is $\sigma = 45\%$ per year. Find the cost of a three-month European put option with a strike price of $95 per share.
Summary:

\[ T = \frac{1}{4}, \quad t = 0, \quad r = 0.08, \]
\[ \sigma = 0.45, \quad S = 97, \quad \text{and} \quad K = 95. \]
Summary:

\[ T = \frac{1}{4}, \quad t = 0, \quad r = 0.08, \]
\[ \sigma = 0.45, \quad S = 97, \quad \text{and} \quad K = 95. \]

\[ w = \ln\left(\frac{S}{K}\right) + \frac{(r + \sigma^2/2)(T - t)}{\sigma \sqrt{T - t}} \]
\[ P = Ke^{-r(T-t)} \phi(\sigma \sqrt{T - t} - w) - S \phi(-w) \]
Example

Summary:

\[ T = \frac{1}{4}, \quad t = 0, \quad r = 0.08, \]
\[ \sigma = 0.45, \quad S = 97, \quad \text{and} \quad K = 95. \]

\[ w = \frac{\ln(S/K) + (r + \sigma^2/2)(T - t)}{\sigma \sqrt{T - t}} \approx 0.293985 \]
\[ P = Ke^{-r(T-t)}\phi(\sigma \sqrt{T - t} - w) - S\phi(-w) \approx 6.71 \]
Each financial firm writing option contracts may have its own estimate of the volatility $\sigma$ of a stock. If we know the price of a call option, its strike price, expiry, the current stock price, and the risk-free interest rate, we can determine the **implied volatility** of the stock.
Each financial firm writing option contracts may have its own estimate of the volatility $\sigma$ of a stock. If we know the price of a call option, its strike price, expiry, the current stock price, and the risk-free interest rate, we can determine the **implied volatility** of the stock.

**Example**

Suppose the current price of a security is $60 per share. The continuously compounded interest rate is 6.25% per year. The cost of a four-month European call option with a strike price of $62 per share is $3. What is the implied volatility of the stock?
We must solve the equation

\[ C = S \phi(w) - Ke^{-rT} \phi(w - \sigma \sqrt{T}) \]

\[ 3 = 60 \phi \left( \frac{0.0625 + \frac{\sigma}{2}}{2} \right) \frac{4}{12} + \ln \frac{60}{62} \]

\[ -62e^{-(0.0625)\frac{4}{12}} \phi \left( \frac{0.0625 + \frac{\sigma}{2}}{2} \right) \frac{4}{12} + \ln \frac{60}{62} - \sigma \sqrt{\frac{4}{12}} \]
Using Newton’s Method, $\sigma \approx 0.241045$. 
The binomial model is a discrete approximation to the Black-Scholes initial value problem originally developed by Cox, Ross, and Rubinstein. **Assumptions:**

- Strike price of the call option is $K$.
- Exercise time of the call option is $T$.
- Present price of the security is $S(0)$.
- Continuously compounded interest rate is $r$.
- Price of the security follows a geometric Brownian motion with variance $\sigma^2$.
- Present time is $t$. 
Binomial Lattice

If the value of the stock is \( S(0) \) then at \( t = T \)

\[
S(T) = \begin{cases} 
  uS(0) \text{ with probability } p, \\
  dS(0) \text{ with probability } 1 - p 
\end{cases}
\]

where \( 0 < d < 1 < u \) and \( 0 < p < 1 \).
Making the Continuous and Discrete Models Agree (1 of 2)

Continuous model:

\[ dS = \mu S \, dt + \sigma S \, dW(t) \]
\[ d(\ln S) = (\mu - \frac{1}{2} \sigma^2) \, dt + \sigma \, dW(t) \]
\[ \mathbb{E}[\ln S(t)] = \ln S(0) + (\mu - \frac{1}{2} \sigma^2) t \]
\[ \text{Var} (\ln S(t)) = \sigma^2 t \]
Continuous model:

\[ dS = \mu S \, dt + \sigma S \, dW(t) \]

\[ d(\ln S) = (\mu - \frac{1}{2} \sigma^2) \, dt + \sigma \, dW(t) \]

\[ E[\ln S(t)] = \ln S(0) + (\mu - \frac{1}{2} \sigma^2) t \]

\[ \text{Var} (\ln S(t)) = \sigma^2 t \]

In the absence of arbitrage \( \mu = r \), i.e. the return on the security should be the same as the return on an equivalent amount in savings.
\[ \ln S(0) + \left( r - \frac{1}{2} \sigma^2 \right) \Delta t = p \ln(uS(0)) + (1 - p) \ln(dS(0)) \]

\[ \left( r - \frac{1}{2} \sigma^2 \right) \Delta t = p \ln u + (1 - p) \ln d \]
\[ \ln S(0) + (r - \frac{1}{2}\sigma^2)\Delta t = p \ln(uS(0)) + (1 - p) \ln(dS(0)) \]

\[ (r - \frac{1}{2}\sigma^2)\Delta t = p \ln u + (1 - p) \ln d \]

The variance in the returns in the continuous and discrete models should also agree.

\[ \sigma^2 \Delta t = p[\ln(uS(0))]^2 + (1 - p)[\ln(dS(0))]^2 - (p \ln(uS(0)) + (1 - p) \ln(dS(0)))^2 = p(1 - p)(\ln u - \ln d)^2 \]
We would like to write $p$, $u$, and $d$ as functions of $r$, $\sigma$, and $\Delta t$.

\[
p \ln u + (1 - p) \ln d = (r - \frac{1}{2} \sigma^2) \Delta t
\]

\[
p(1 - p)(\ln u - \ln d)^2 = \sigma^2 \Delta t
\]
We would like to write $p$, $u$, and $d$ as functions of $r$, $\sigma$, and $\Delta t$.

\[
p \ln u + (1 - p) \ln d = (r - \frac{1}{2} \sigma^2) \Delta t
\]

\[
p(1 - p) (\ln u - \ln d)^2 = \sigma^2 \Delta t
\]

- We need a third equation in order to solve this system.
We would like to write $p$, $u$, and $d$ as functions of $r$, $\sigma$, and $\Delta t$.

\[
\begin{align*}
    p \ln u + (1 - p) \ln d &= (r - \frac{1}{2} \sigma^2) \Delta t \\
    p(1 - p) (\ln u - \ln d)^2 &= \sigma^2 \Delta t
\end{align*}
\]

- We need a third equation in order to solve this system.
- We are free to pick any equation consistent with the first two.
We would like to write $p$, $u$, and $d$ as functions of $r$, $\sigma$, and $\Delta t$.

$$p \ln u + (1 - p) \ln d = (r - \frac{1}{2} \sigma^2) \Delta t$$

$$p(1 - p) (\ln u - \ln d)^2 = \sigma^2 \Delta t$$

- We need a third equation in order to solve this system.
- We are free to pick any equation consistent with the first two.
- We pick $d = 1/u$ (why?).
Solving the System

\[(2p - 1) \ln u = (r - \frac{1}{2}\sigma^2)\Delta t\]

\[4p(1 - p)(\ln u)^2 = \sigma^2\Delta t\]

1. Square the first equation and add to the second.
2. Ignore terms involving \((\Delta t)^2\).
Solving the System

\[
(2p - 1) \ln u = \left( r - \frac{1}{2} \sigma^2 \right) \Delta t
\]

\[
4p(1 - p)(\ln u)^2 = \sigma^2 \Delta t
\]

1. Square the first equation and add to the second.
2. Ignore terms involving \((\Delta t)^2\).

\[
u = e^{\sigma \sqrt{\Delta t}}
\]

\[
d = e^{-\sigma \sqrt{\Delta t}}
\]

\[
p = \frac{1}{2} \left( 1 + \left( \frac{r}{\sigma} - \frac{\sigma}{2} \right) \sqrt{\Delta t} \right)
\]
Example

Suppose \( S(0) = 1, r = 0.10, \sigma = 0.20, T = 1/4, \Delta t = 1/12 \), then the lattice of security prices resembles:

\[
\begin{array}{c c c c}
1.18911 & & & \\
1.1224 & & & \\
1.05943 & 1.05943 & & \\
0.9439 & 0.9439 & 0.890947 & \\
0.890947 & 0.840965 & \\
\end{array}
\]
Determining a European Call Price

Payoff: \((S(T) - K)^+\)
Let \(Y\) be a binomial random variable with probability of an UP step \(p\) and \(n\) total steps.

\[
C = e^{-rT} E \left[ (u^Y d^{n-Y} S(0) - K)^+ \right] \\
= e^{-rT} E \left[ (e^{Y \sigma \sqrt{\Delta t}} e^{(Y-n)\sigma \sqrt{\Delta t}} S(0) - K)^+ \right] \\
= e^{-rT} E \left[ (e^{(2Y-n)\sigma \sqrt{\Delta t}} S(0) - K)^+ \right] \\
= e^{-rT} E \left[ (e^{(2Y-T/\Delta t)\sigma \sqrt{\Delta t}} S(0) - K)^+ \right].
\]
The price of a security is $62, the continuously compounded interest rate is 10% per year, the volatility of the price of the security is $\sigma = 20\%$ per year. If the strike price of a call option is $60$ per share with an expiry of 5 months, then $C = 5.789$ according to the solution to the Black-Scholes equation.

The parameters of the discrete model are:

\[ u = 1.05943, \quad d = 0.9439, \quad \text{and} \quad p = 0.557735. \]
Lattice of Security Prices
Payoffs of the Call Option

<table>
<thead>
<tr>
<th>$S$</th>
<th>$(S - K)^+$</th>
<th>Prob.</th>
</tr>
</thead>
<tbody>
<tr>
<td>82.7488</td>
<td>22.7488</td>
<td>$(\frac{5}{5})u^5d^0 \approx 0.0539684$</td>
</tr>
<tr>
<td>73.7248</td>
<td>13.7248</td>
<td>$(\frac{4}{5})u^4d^1 \approx 0.213976$</td>
</tr>
<tr>
<td>65.6849</td>
<td>5.6849</td>
<td>$(\frac{3}{5})u^3d^2 \approx 0.339351$</td>
</tr>
<tr>
<td>58.5218</td>
<td>0</td>
<td>$(\frac{2}{5})u^2d^3 \approx 0.269094$</td>
</tr>
<tr>
<td>52.1398</td>
<td>0</td>
<td>$(\frac{1}{5})u^1d^3 \approx 0.106691$</td>
</tr>
<tr>
<td>46.4538</td>
<td>0</td>
<td>$(\frac{0}{5})u^0d^5 \approx 0.0169205$</td>
</tr>
</tbody>
</table>

$$C \approx \frac{(5.6849)(0.3394) + (13.7248)(0.2140) + (22.7488)(0.0540)}{e^{(0.10)(5/12)}}$$
$$= 5.83509.$$