Brownian Motion

An Undergraduate Introduction to Financial Mathematics

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Background

We have already seen that the limiting behavior of a **discrete random walk** yields a derivation of the **normal probability density function.**

Today we explore some further properties of the discrete random walk and introduce the concept of **stochastic processes.**

First Step Analysis

Assumptions:

- Current value of security is $S(0)$.
- At each "tick" of a clock S may change by ± 1.
- $\mathrm{P}\left(S(n+1) = S(n) + 1\right) = 1/2$ and $\mathrm{P}\left(S(n+1) = S(n) - 1\right) = 1/2$.

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If $X_i = \begin{cases} +1 & \text{with probability } 1/2, \\ -1 & \text{with probability } 1/2 \end{cases}$ then

$$S(N) = S(0) + X_1 + X_2 + \cdots + X_N$$

Further Assumptions

- X_i and X_j are independent when $i \neq j$.
- Out of n random selections $X = +1$, k times.

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Then

$$S(n) = S(0) + k - (n - k) = S(0) + 2k - n$$

and

$$\mathrm{P}\left(S(n) = S(0) + 2k - n\right) = \binom{n}{k}\left(\frac{1}{2}\right)^n.$$

Spatial Homogeneity

Define $T(i) = S(i) - S(0)$ for $i = 0, 1, \ldots, n$ then

- $T(0) = 0$,
- $S(n) = S(0) + 2k - n$ if and only if $T(n) = 2k - n$.

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Question: what states can be visited in n steps?

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Lemma

For the random walk $S(i)$ with initial state $S(0) = 0$,

1. $\mathrm{P}\left(S(n) = m\right) = 0$ *if* $|m| > n$,
2. $\mathrm{P}\left(S(n) = m\right) = 0$ *if* $n + m$ *is odd,*
3. $\mathrm{P}\left(S(n) = m\right) = \binom{n}{(n+m)/2} \left(\frac{1}{2}\right)^n$, *otherwise.*

Mean and Variance of Random Walk (1 of 2)

Theorem

For the random walk $S(i)$ with initial state $S(0) = 0$,

$$\mathrm{E}\left[S(n)\right] = 0 \qquad \textit{and} \qquad \mathrm{Var}\left(S(n)\right) = n.$$

Mean and Variance of Random Walk (2 of 2)

Proof.

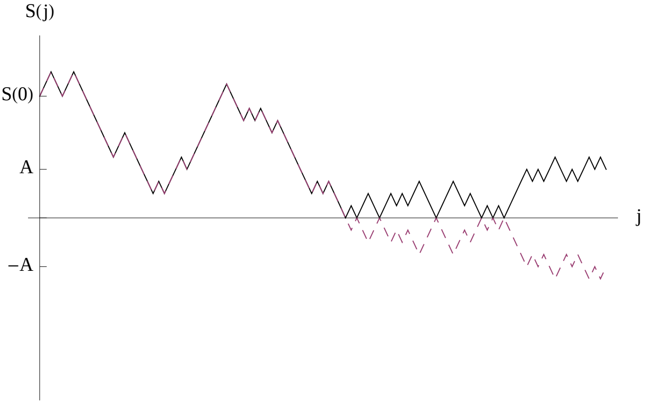
$$\begin{aligned} \mathrm{E}\left[S(n)\right] &= \mathrm{E}\left[S(0)\right] + \mathrm{E}\left[X_1\right] + \mathrm{E}\left[X_2\right] + \cdots + \mathrm{E}\left[X_n\right] \\ &= 0 \end{aligned}$$

since $\mathrm{E}\left[X_i\right] = 0$ for $i = 1, 2, \ldots, n$. If X_i and X_j are independent when $i \neq j$ we have

$$\mathrm{Var}\left(S(n)\right) = \mathrm{Var}\left(S(0)\right) + \sum_{i=1}^{n} \mathrm{Var}\left(X_i\right) = n.$$

□

Consider a random walk for which $S(k) = 0$ for some k.



Reflections of Random Walks (2 of 2)

If $S(k) = 0$ then define another random walk $\hat{S}(j)$ by

$$\hat{S}(j) = \begin{cases} S(j) & \text{for } j = 0, 1, \ldots, k \\ -S(j) & \text{for } j = k+1, k+2, \ldots, n. \end{cases}$$

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Since UP/DOWN steps occur with equal probability,

$$\mathrm{P}\left(S(n) = A\right) = \mathrm{P}\left(\hat{S}(n) = -A\right).$$

Markov Property

Random walks have no "memory" of how they arrive at a particular state. Only the current state influences the next state.

$$
\begin{aligned}
\mathrm{P}\left(S(n) = A\right) &= \mathrm{P}\left(\hat{S}(n) = -A\right) \\
\mathrm{P}\left(S(k) = 0\right)\mathrm{P}\left(T(n-k) = A\right) &= \mathrm{P}\left(\hat{S}(k) = 0\right)\mathrm{P}\left(\hat{T}(n-k) = -A\right) \\
&= \mathrm{P}\left(S(k) = 0\right)\mathrm{P}\left(\hat{T}(n-k) = -A\right) \\
\mathrm{P}\left(T(n-k) = A\right) &= \mathrm{P}\left(\hat{T}(n-k) = -A\right)
\end{aligned}
$$

A Result

Theorem

If $\{S(j)\}_{j=0}^{n}$ *is an unbiased random walk with initial state* $S(0) = i$ *and if* $|A - i| \leq n$ *and* $|A + i| \leq n$ *then*

$$\mathrm{P}\left(S(n) = A \,|\, S(0) = i\right) = \mathrm{P}\left(S(n) = -A \,|\, S(0) = i\right).$$

A Result

Theorem

If $\{S(j)\}_{j=0}^{n}$ is an unbiased random walk with initial state $S(0) = i$ and if $|A - i| \leq n$ and $|A + i| \leq n$ then

$$\mathrm{P}\left(S(n) = A \,|\, S(0) = i\right) = \mathrm{P}\left(S(n) = -A \,|\, S(0) = i\right).$$

These probabilities are 0 if $n + A - i$ is odd (and consequently $n - A - i$ is odd).

Absorbing Boundary Conditions

Remark: so far we have considered only random walks which were free to wander unrestricted.

What if there is a state A such that if $S(k) = A$ then $S(n) = A$ for all $n \geq k$? Such a state is called an **absorbing boundary condition.**

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Example

A gambler going broke and unable to borrow money has encountered an absorbing boundary condition.

Questions

Suppose random walk $S(i)$ has an absorbing boundary condition at 0. If $0 < S(0) < A$,

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Suppose random walk $S(i)$ has an absorbing boundary condition at 0. If $0 < S(0) < A$,

1. what is the probability that the state of the random walk crosses the threshold value of A before it hits the boundary at 0?
2. what is the expected value of the number of steps which will elapse before the state of the random variable first crosses the A threshold?

Define $S_{\min}(n) = \min\{S(k) : 0 \leq k \leq n\}$ which can be thought of as the smallest value the random walk takes on.

The probability the state of the random walk crosses the threshold value of A before it hits the boundary at 0 is then

$$\mathrm{P}\left(S(n) = A \wedge S_{\min}(n) > 0 \,|\, S(0) = i\right).$$

Answer to First Question (2 of 2)

Lemma

Suppose a random walk $S(k) = S(0) + \sum_{i=1}^{k} X_i$ in which the X_i for $i = 1, 2, \ldots$ are independent, identically distributed random variables taking on the values ± 1, each with probability $p = 1/2$. Suppose further that the boundary at 0 is absorbing, then if $A, i > 0$,

$$\begin{aligned} f_{A,i}(n) &= \mathrm{P}\left(S(n) = A \wedge S_{\min}(n) > 0 \mid S(0) = i\right) \\ &= \left[\binom{n}{(n+A-i)/2} - \binom{n}{(n-A-i)/2}\right]\left(\frac{1}{2}\right)^n, \end{aligned}$$

provided $|A - i| \leq n$, $|A + i| \leq n$, and $n + A - i$ is even.

Consider a random walk with no boundary, that is, the random variable $S(n)$ has an initial state of $S(0) = i > 0$ and $S(k)$ is allowed to wander into negative territory (and back) arbitrarily. In this situation

$$\begin{aligned} & \mathrm{P}\left(S(n) = A \,|\, S(0) = i\right) \\ = \quad & \mathrm{P}\left(S(n) = A \wedge S_{\min}(n) > 0 \,|\, S(0) = i\right) \\ & + \mathrm{P}\left(S(n) = A \wedge S_{\min}(n) \leq 0 \,|\, S(0) = i\right) \end{aligned}$$

by the Addition Rule.

Proof (2 of 3)

Now consider the probability on the left-hand side of the equation.

$$\mathrm{P}\left(S(n) = A \,|\, S(0) = i\right)$$

It possesses no boundary condition and by the spatial homogeneity of the random walk

$$\mathrm{P}\left(S(n) = A \,|\, S(0) = i\right) = \mathrm{P}\left(T(n) = A - i\right)$$

where $\{T(j)\}_{j=0}^{n}$ is an unbiased random walk with initial state $T(0) = 0$. Hence $\mathrm{P}\left(T(n) = A - i\right) = 0$ unless $n + A - i$ is even and $|A - i| \leq n$, in which case

$$\mathrm{P}\left(S(n) = A \,|\, S(0) = i\right) = \binom{n}{(n + A - i)/2} \left(\frac{1}{2}\right)^{n}.$$

On the other hand if the random walk starts at a positive state i and finishes at $-A < 0$ then it is certain that $S_{\min}(n) \leq 0$. Consequently

$$\begin{aligned} \mathrm{P}\left(S(n) = A \wedge S_{\min}(n) \leq 0 \,|\, S(0) = i\right) &= \mathrm{P}\left(S(n) = -A \,|\, S(0) = i\right) \\ &= \binom{n}{(n-A-i)/2}\left(\frac{1}{2}\right)^n \end{aligned}$$

provided $|A + i| \leq n$ and $n - A - i$ is even. Finally

$$\begin{aligned} &\mathrm{P}\left(S(n) = A \wedge S_{\min}(n) > 0 \,|\, S(0) = i\right) \\ &= \binom{n}{(n+A-i)/2}\left(\frac{1}{2}\right)^n - \binom{n}{(n-A-i)/2}\left(\frac{1}{2}\right)^n. \end{aligned}$$

Example

For an unbiased random walk with initial state $S(0) = 10$, what is the probability that $S(50) = 16$ and $S(n) > 0$ for $n = 0, 1, \ldots, 50$?

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$$
\begin{aligned}
f_{16,10}(50) &= \mathrm{P}\left(S(50) = 16 \wedge m_{50} > 0 \,|\, S(0) = 10\right) \\
&= \left[\binom{50}{28} - \binom{50}{12}\right] 2^{-50} \\
&\approx 0.0787178
\end{aligned}
$$

Stopping Times

Define $\Omega_A = \min\{n \,|\, S(n) = A\}$ which is the first time that the random walk $S(n) = A$. This is called the **stopping time.**

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Suppose $A = 0$, then $\Omega_0 = n$ if and only if $S(0) = i > 0$, $S(n-1) = 1$, $m_{n-1} > 0$ and $X_n = -1$.

$$\begin{aligned} & \mathrm{P}\left(\Omega_0 = n \,|\, S(0) = i\right) \\ &= \mathrm{P}\left(X_n = -1 \wedge S(n-1) = 1 \wedge m_{n-1} > 0 \,|\, S(0) = i\right) \\ &= \frac{1}{2}\mathrm{P}\left(S(n-1) = 1 \wedge m_{n-1} > 0 \,|\, S(0) = i\right) \\ &= \frac{1}{2} f_{1,i}(n-1). \end{aligned}$$

Thus by spatial homogeneity

$$\mathrm{P}\left(\Omega_A = n \,|\, S(0) = i\right) = \frac{1}{2} f_{1,(i-A)}(n-1)$$

Paths

We can analyze the stopping time by think of the random walk as having two boundaries, one at 0 and another at A.

$p_{i \to A}$: any random walk $\{S(j)\}$ in the discrete interval $[0, A]$ starting at $i > 0$, terminating at A, and which avoids 0.

$P_{p_{i \to A}}$: the probability that the random walk starting at $S(0) = i$ follows $p_{i \to A}$.

$\mathcal{P}_A(i)$: the probability that a random walk which starts at $S(0) = i$ will achieve state $S = A$ while avoiding the state $S = 0$.

Determination of $\mathcal{P}_A(i)$

$$\begin{aligned}
\mathcal{P}_A(i) &= \sum_{p_{i\to A}} P_{p_{i\to A}} \\
&= \mathrm{P}\left(S(1) = i-1 \mid S(0) = i\right) \mathcal{P}_A(i-1) \\
&\quad + \mathrm{P}\left(S(1) = i+1 \mid S(0) = i\right) \mathcal{P}_A(i+1) \\
&= \frac{1}{2}\mathcal{P}_A(i-1) + \frac{1}{2}\mathcal{P}_A(i+1)
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&= \frac{1}{2}\mathcal{P}_A(i-1) + \frac{1}{2}\mathcal{P}_A(i+1)
\end{aligned}
$$

This implies

$$
\mathcal{P}_A(i-1) - 2\mathcal{P}_A(i) + \mathcal{P}_A(i+1) = 0.
$$

Probability of Exiting

Theorem

Suppose $S(k) = S(0) + \sum_{i=1}^{k} X_i$ *where the* X_i *for* $i = 1, 2, \ldots$ *are independent, identically distributed random variables taking on the values* ± 1*, each with probability* $p = 1/2$*. Suppose further that the boundaries at* 0 *and* A *are absorbing, then if* $0 \leq S(0) = i \leq A$

1. *the probability that the random walk achieves state* A *without achieving state* 0 *is* $\mathcal{P}_A(i) = i/A$*,*
2. *the probability that the random walk achieves state* 0 *without achieving state* A *is* $\mathcal{P}_0(i) = 1 - i/A$*.*

Proof

Suppose $\mathcal{P}_A(i) = \alpha + \beta i$ where α and β are constants. Substituting into the difference equation yields

$$\alpha + \beta(i-1) - 2(\alpha + \beta i) + \alpha + \beta(i+1) = 0$$

so $\mathcal{P}_A(i)$ solves the difference equation.

Since $\mathcal{P}_A(0) = 0$, then $\alpha = 0$.

Since $\mathbf{P}_A(A) = 1$, then $\beta = 1/A$.

Consequently $\mathcal{P}_A(i) = i/A$.

Simpler Question

Question: what is the expected exit time through either boundary $A > 0$ or 0?

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We make the following definitions:

- B: the set of boundary points, $B = \{0, A\}$.
- $\omega_{p_{i \to B}}$: the exit time of the random walk which starts at $S(0) = i$, where $0 \leq i \leq A$ and which follows path $p_{i \to B}$.
- $\Omega_B(i)$: the expected value of the exit time for a random walk which starts at $S(0) = i$, where $0 \leq i \leq A$.

Simple Answer

$$\begin{aligned}\Omega_B(i) &= \sum_{p_{i\to B}} P_{p_{i\to B}} \omega_{p_{i\to B}} \\ &= \frac{1}{2}(1+\Omega_B(i-1)) + \frac{1}{2}(1+\Omega_B(i+1))\end{aligned}$$

Since the path from $i \to B$ can be decomposed into paths from $(i-1) \to B$ and $(i+1) \to B$ with the addition of a single step, the expected value of the exit time of a random walk starting at i is one more than the expected value of a random walk starting at $i \pm 1$.

System of Equations

$$\Omega_B(i-1) - 2\Omega_B(i) + \Omega_B(i+1) = -2$$

for $i = 1, 2, \dots, A-1$, while $\Omega_B(0) = 0 = \Omega_B(A)$.

System of Equations

$$\Omega_B(i-1) - 2\Omega_B(i) + \Omega_B(i+1) = -2$$

for $i = 1, 2, \dots, A-1$, while $\Omega_B(0) = 0 = \Omega_B(A)$.

Try a solution of the form $\Omega_B(i) = ai^2 + bi + c$ and determine the coefficients a, b, and c.

A Result

Theorem

Suppose $S(k) = S(0) + \sum_{i=1}^{k} X_i$ *where the* X_i *for* $i = 1, 2, \dots$ *are independent, identically distributed random variables taking on the values* ± 1*, each with probability* $p = 1/2$*. Suppose further that the boundaries at* 0 *and* A *are absorbing, then if* $0 \leq S(0) = i \leq A$ *the random walk intersects the boundary (*$S = 0$ *or* $S = A$*) after a mean number of steps given by the formula*

$$\Omega_B(i) = i(A - i).$$

Example

Suppose an unbiased random walk takes place on the discrete interval $\{0, 1, 2, \ldots, 10\}$ for which the boundaries at 0 and 10 are absorbing. As a function of the initial condition *i*, find the expected value of the exit time.

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i	0	1	2	3	4	5	6	7	8	9	10
$\Omega_B(i)$	0	9	16	21	24	25	24	21	16	9	0

Main Question: Conditional Exit Time

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$$\Omega_A(i) = \frac{\displaystyle\sum_{p_{i\to A}} P_{p_{i\to A}} \omega_{p_{i\to A}}}{\displaystyle\sum_{p_{i\to A}} P_{p_{i\to A}}} = \frac{\displaystyle\sum_{p_{i\to A}} P_{p_{i\to A}} \omega_{p_{i\to A}}}{\mathcal{P}_A(i)}$$

Decomposing the Walk (1 of 2)

$$
\begin{aligned}
\Omega_A(i) &= 1 + \frac{\frac{1}{2}\Omega_A(i-1)\mathcal{P}_A(i-1) + \frac{1}{2}\Omega_A(i+1)\mathcal{P}_A(i+1)}{\mathcal{P}_A(i)} \\
&= \mathcal{P}_A(i) + \frac{1}{2}\Omega_A(i-1)\mathcal{P}_A(i-1) + \frac{1}{2}\Omega_A(i+1)\mathcal{P}_A(i+1) \\
\Omega_A(i)\frac{i}{A} &= \frac{i}{A} + \frac{i-1}{2A}\Omega_A(i-1) + \frac{i+1}{2A}\Omega_A(i+1) \\
2i\Omega_A(i) &= 2i + (i-1)\Omega_A(i-1) + (i+1)\Omega_A(i+1)
\end{aligned}
$$

Decomposing the Walk (2 of 2)

The last equation is equivalent to

$$(i-1)\Omega_A(i-1) - 2i\Omega_A(i) + (i+1)\Omega_A(i+1) = -2i.$$

Decomposing the Walk (2 of 2)

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Assuming $\Omega_A(i) = ai^2 + bi + c$, determine the coefficients a, b, and c.

Conditional Exit Time

Theorem

Suppose $S(k) = S(0) + \sum_{i=1}^{k} X_i$ *where the* X_i *for* $i = 1, 2, \ldots$ *are independent, identically distributed random variables taking on the values* ± 1*, each with probability* $p = 1/2$*. Suppose further that the boundary at* 0 *is absorbing. The random walk that avoids state* 0 *will stop the first time that* $S(n) = A$*. The expected value of the stopping time is*

$$\Omega_A(i) = \frac{1}{3}\left(A^2 - i^2\right), \quad \text{for } i = 1, 2, \ldots, A.$$

Remark: If the random walk starts in state 0, since this state is absorbing the expected value of the exit time is infinity.

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i	1	2	3	4	5	6	7	8	9	10
$\Omega_{10}(i)$	33	32	$\frac{91}{3}$	28	25	$\frac{64}{3}$	17	12	$\frac{19}{3}$	0

Now we begin the development of *continuous* random walks by taking a limit of the previous discrete random walks.

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Assumptions:

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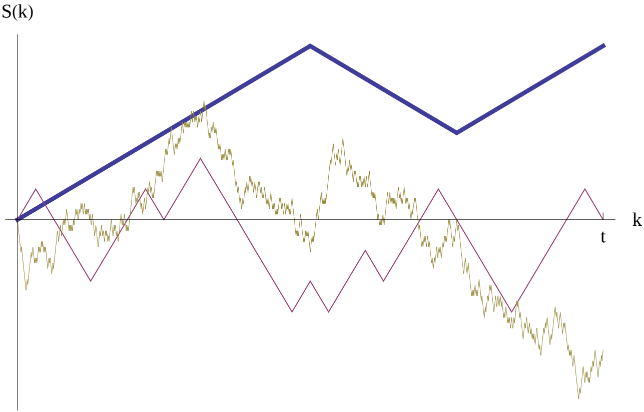
Find $\mathrm{E}\left[S(t)\right]$ and $\mathrm{Var}\left(S(t)\right)$.

Illustration



Brownian Motion/Wiener Process

The continuous limit of this random walk is denoted $W(t)$ and is called a **Wiener process.**

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The continuous limit of this random walk is denoted $W(t)$ and is called a **Wiener process.**

1. $W(t)$ is a continuous function of t,
2. $W(0) = 0$ with probability one,
3. Spatial homogeneity: if $W_0(t)$ represents a Wiener process for which the initial state is 0 and if $W_x(t)$ represents a Wiener process for which the initial state is x, then $W_x(t) = x + W_0(t)$.
4. Markov property: for $0 < s < t$ the conditional distribution of $W(t)$ depends on the value of $W(s) + W(t - s)$.
5. For each t, $W(t)$ is normally distributed with mean zero and variance t,
6. The changes in W in non-overlapping intervals of t are independent random variables with means of zero and variances equal to the lengths of the time intervals.

More Properties

Suppose $0 \leq t_1 < t_2$ and define $\Delta W_{[t_1,t_2]} = W(t_2) - W(t_1)$.

$$
\begin{aligned}
\mathrm{Var}\left(\Delta W_{[t_1,t_2]}\right) &= \mathrm{E}\left[(W(t_2) - W(t_1))^2\right] - \mathrm{E}\left[W(t_2) - W(t_1)\right]^2 \\
&= \mathrm{E}\left[(W(t_2))^2\right] + \mathrm{E}\left[(W(t_1))^2\right] - 2\mathrm{E}\left[W(t_1)W(t_2)\right] \\
&= t_2 + t_1 - 2\mathrm{E}\left[W(t_1)(W(t_2) - W(t_1) + W(t_1))\right] \\
&= t_2 + t_1 - 2\mathrm{E}\left[W(t_1)(W(t_2) - W(t_1))\right] \\
&\quad - 2\mathrm{E}\left[(W(t_1))^2\right] \\
&= t_2 + t_1 - 2t_1 \\
&= t_2 - t_1.
\end{aligned}
$$

Differential Wiener Process

We have seen that for $0 \leq t_1 < t_2$,

$$\mathrm{Var}\left(\Delta W\right) = \mathrm{E}\left[(\Delta W)^2\right] = \Delta t.$$

This is also true in the limit as Δt becomes small, thus we write

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$$(dW(t))^2 = dt.$$

Theorem

The derivative dW/dt does not exist for any t.

Proof

Recall the limit definition of the derivative from calculus,

$$\frac{df}{dt} = \lim_{h \to 0} \frac{f(s+h) - f(s)}{h}.$$

Suppose $f(t)$ is a Wiener process $W(t)$. Since

$$\mathrm{E}\left[(W(s+h) - W(s))^2\right] = \mathrm{E}\left[|W(s+h) - W(s)|^2\right] = h$$

then on average $|W(s+h) - W(s)| \approx \sqrt{h}$, and therefore

$$\lim_{h \to 0} \frac{W(s+h) - W(s)}{h} \quad \text{does not exist.}$$

Integral of a Wiener Process

The **stochastic integral** of $f(x)$ on the interval $[0, t]$ is defined to be

$$\begin{aligned} Z(t) - Z(0) &= \int_0^t f(\tau)\, dW(\tau) \\ &= \lim_{n \to \infty} \sum_{k=1}^{n} f(t_{k-1}) \left(W(t_k) - W(t_{k-1}) \right) \end{aligned}$$

where $t_k = kt/n$.

Note: The function f is evaluated at the left-hand endpoint of each subinterval.

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Note: The function f is evaluated at the left-hand endpoint of each subinterval.

The stochastic integral is equivalent to its **differential form**

$$dZ = f(t)\, dW(t)$$

ODE: Exponential Growth

If $P(0) = P_0$ and the rate of change of P is proportional to P, then

$$\frac{dP}{dt} = \mu P,$$

and $P(t) = P_0 e^{\mu t}$.

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If we let $Z = \ln P$ then the ODE becomes

$$dZ = \mu \, dt.$$

Stochastic Differential Equation (SDE)

Perturb dZ by adding a Wiener process with mean zero and standard deviation $\sigma\sqrt{dt}$.

$$dZ = \mu\, dt + \sigma\, dW(t)$$

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$$dZ = \mu\, dt + \sigma\, dW(t)$$

This is a **generalized Wiener process.** The constant μ is called the **drift** and the constant σ is called the **volatility.** The solution to the SDE is

$$Z(t) = Z(0) + \mu t + \int_0^t \sigma\, dW(\tau).$$

Expectation and Variance

$$\begin{aligned} \mathrm{E}\left[Z(t) - Z(0)\right] &= \mu t \\ \mathrm{Var}\left(Z(t) - Z(0)\right) &= \sigma^2 t \end{aligned}$$

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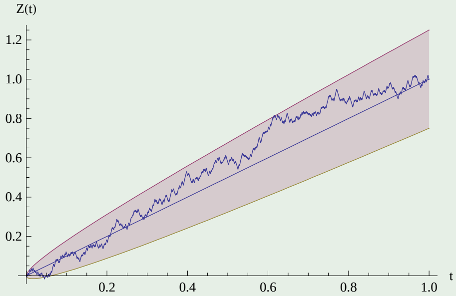
In terms of numerical approximation,

$$\int_0^t dW(\tau) \approx \sum_{j=1}^{n} X_j$$

where X_j is a normal random variable with mean 0 and variance t/n.

Example

Suppose the drift parameter is $\mu = 1$ and the volatility is $\sigma = 1/4$, then the expected value of the Wiener process is t and the standard deviation is $\sqrt{t}/4$.



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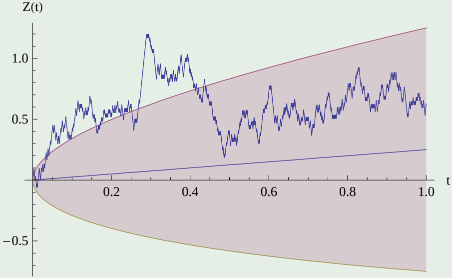

Simple Generalization

If the drift and volatility are functions of t then

$$dZ = \mu(t)\, dt + \sigma(t)\, dW(t).$$

and

$$Z(t) = Z(0) + \int_0^t \mu(\tau)\, d\tau + \int_0^t \sigma(\tau)\, dW(\tau).$$

Itô Processes

A stochastic process of the form

$$dS = a(S,t)\,dt + b(S,t)\,dW(t)$$

is called an **Itô process.**

We will shortly be called upon to develop new stochastic processes which are functions of S. Suppose $Z = \ln S$, then $dZ = dS/S$ (by the chain rule), but are the following two stochastic processes equivalent?

$$\begin{aligned} dS &= \mu S\,dt + \sigma S\,dW(t) \\ dZ &= \mu\,dt + \sigma\,dW(t) \end{aligned}$$

Discussion

$$
\begin{aligned}
dS &= \mu S\, dt + \sigma S\, dW(t) \\
dZ &= \mu\, dt + \sigma\, dW(t)
\end{aligned}
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- $\mu S \to 0$ and $\sigma S \to 0$ as $S \to 0^+$.

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- First equation makes a suitable mathematical model for a stock price $S \geq 0$, in second equation Z could go negative.

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- Second equation can be integrated, first cannot.

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- $\mu S \to 0$ and $\sigma S \to 0$ as $S \to 0^+$.
- First equation makes a suitable mathematical model for a stock price $S \geq 0$, in second equation Z could go negative.
- Second equation can be integrated, first cannot.
- The two equations are not equivalent because the chain rule does not apply to functions of stochastic quantities.

Lemma (Itô's Lemma)

Suppose that the random variable X is described by the Itô process

$$dX = a(X,t)\,dt + b(X,t)\,dW(t)$$

where $dW(t)$ is a normal random variable. Suppose the random variable $Y = F(X,t)$. Then Y is described by the following Itô process.

$$dY = \left(a(X,t)F_X + F_t + \frac{1}{2}(b(X,t))^2 F_{XX} \right) dt + b(X,t)F_X dW(t)$$

Multivariable Form of Taylor's Theorem (1 of 3)

If $f(x)$ is an $(n+1)$-times differentiable function on an open interval containing x_0 then the function may be written as

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(\theta)}{(n+1)!}(x - x_0)^{n+1} \qquad (1)$$

The last term above is usually called the Taylor remainder formula and is denoted by R_{n+1}. The quantity θ lies between x and x_0. The other terms form a polynomial in x of degree at most n and can be used as an approximation for $f(x)$ in a neighborhood of x_0.

Multivariable Form of Taylor's Theorem (2 of 3)

Suppose the function $F(y, z)$ has partial derivatives up to order three on an open disk containing the point with coordinates (y_0, z_0). Define the function $f(x) = F(y_0 + xh, z_0 + xk)$ where h and k are chosen small enough that $(y_0 + h, z_0 + k)$ lie within the disk surrounding (y_0, z_0). Since f is a function of a single variable then we can use the single-variable form of Taylor's formula in Eq. (1) with $x_0 = 0$ and $x = 1$ to write

$$f(1) = f(0) + f'(0) + \frac{1}{2}f''(0) + R_3. \tag{2}$$

Using the multivariable chain rule for derivatives we have, upon differentiating $f(x)$ and setting $x = 0$,

$$f'(0) = hF_y(y_0, z_0) + kF_z(y_0, z_0) \tag{3}$$

$$f''(0) = h^2 F_{yy}(y_0, z_0) + 2hkF_{yz}(y_0, z_0) + k^2 F_{zz}(y_0, z_0). \tag{4}$$

We have made use of the fact that $F_{yz} = F_{zy}$ for this function under the smoothness assumptions. The remainder term R_3 contains only third order partial derivatives of F evaluated somewhere on the line connecting the points (y_0, z_0) and $(y_0 + h, z_0 + k)$. Thus if we substitute Eqs. (3) and (4) into (2) we obtain

$$\begin{aligned}
\Delta F &= f(1) - f(0) \qquad (5)\\
&= F(y_0 + h, z_0 + k) - F(y_0, z_0)\\
&= R_3 + hF_y(y_0, z_0) + kF_z(y_0, z_0)\\
&\quad + \frac{1}{2}\left(h^2 F_{yy}(y_0, z_0) + 2hkF_{yz}(y_0, z_0) + k^2 F_{zz}(y_0, z_0)\right).
\end{aligned}$$

This last equation can be used to derive Itô's Lemma.

Proof (1 of 3)

Let X be a random variable described by an Itô process of the form

$$dX = a(X,t)\,dt + b(X,t)\,dW(t) \tag{6}$$

where $dW(t)$ is a normal random variable and a and b are functions of X and t. Let $Y = F(X,t)$ be another random variable defined as a function of X and t. Given the Itô process which describes X we will now determine the Itô process which describes Y.

Proof (2 of 3)

Using a Taylor series expansion for Y detailed in (5) we find

$$\begin{aligned}
\Delta Y &= F_X \Delta X + F_t \Delta t + \frac{1}{2} F_{XX} (\Delta X)^2 + F_{Xt} \Delta X \Delta t \\
&\quad + \frac{1}{2} F_{tt} (\Delta t)^2 + R_3 \\
&= F_X (a \Delta t + b\, dW(t)) + F_t \Delta t + \frac{1}{2} F_{XX} (a \Delta t + b\, dW(t))^2 \\
&\quad + F_{Xt} (a \Delta t + b\, dW(t)) \Delta t + \frac{1}{2} F_{tt} (\Delta t)^2 + R_3 .
\end{aligned}$$

Proof (3 of 3)

Upon simplifying, the expression ΔX has been replaced by the discrete version of the Itô process. Thus as Δt becomes small

$$\Delta Y \approx F_X(a\,dt + b\,dW(t)) + F_t\,dt + \frac{1}{2!}F_{XX}b^2(dW(t))^2.$$

Using the relationship $(dW(t))^2 = dt$

$$\Delta Y \approx F_X(a\,dt + b\,dW(t)) + F_t\,dt + \frac{1}{2!}F_{XX}b^2\,dt. \tag{7}$$

Example

If $Z = \ln S$ and

$$dS = \mu S\, dt + \sigma S\, dW(t),$$

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If $Z = \ln S$ then

$$\begin{aligned} dZ &= \left(\mu S\left[\frac{1}{S}\right] + 0 + \frac{1}{2}\sigma^2 S^2\left[-\frac{1}{S^2}\right]\right)\,dt + \sigma S\left(\frac{1}{S}\right)\,dW(t) \\ &= \left(\mu - \frac{\sigma^2}{2}\right)\,dt + \sigma\,dW(t) \end{aligned}$$

Example

If $S = e^Z$ and

$$dZ = \mu\, dt + \sigma\, dW(t),$$

find the stochastic process followed by S.

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If $S = e^Z$ and

$$dZ = \mu\, dt + \sigma\, dW(t),$$

find the stochastic process followed by S.

If $S = e^Z$ then

$$\begin{aligned} dS &= \left(\mu\left[e^Z\right] + 0 + \frac{1}{2}\sigma^2\left[e^Z\right]\right) dt + \sigma\left(e^Z\right) dW(t) \\ &= \left(\mu + \frac{\sigma^2}{2}\right) S\, dt + \sigma S\, dW(t) \end{aligned}$$

Stock Example (1 of 2)

- Suppose we collect stock prices for $n+1$ days: $\{S(0), S(1), \ldots, S(n)\}$.

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Stock Example (1 of 2)

- Suppose we collect stock prices for $n + 1$ days: $\{S(0), S(1), \ldots, S(n)\}$.
- Under the lognormal assumption $Z(i) = \ln S(i+1)/S(i)$ is a normal random variable.
- If the mean (drift) and variance (volatility squared) of Z are μ and σ^2 respectively, then

$$dZ = \mu\, dt + \sigma\, dW(t).$$

Stock Example (2 of 2)

Hence

$$Z(t) = Z(0) + \mu t + \int_0^t \sigma \, dW(\tau)$$

and

$$S(t) = S(0) e^{\mu t + \int_0^t \sigma \, dW(\tau)}.$$

Stock Example (2 of 2)

Hence

$$Z(t) = Z(0) + \mu t + \int_0^t \sigma \, dW(\tau)$$

and

$$S(t) = S(0) e^{\mu t + \int_0^t \sigma \, dW(\tau)}.$$

The mean and variance of $S(t)$ are

$$\begin{aligned} \mathrm{E}\left[S(t)\right] &= S(0) e^{(\mu + \sigma^2/2)t} \\ \mathrm{Var}\left(S(t)\right) &= (S(0))^2 e^{(2\mu + \sigma^2)t} \left(e^{\sigma^2 t} - 1\right). \end{aligned}$$