Extensions to the Black-Scholes Equation

An Undergraduate Introduction to Financial Mathematics

J. Robert Buchanan

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We have versions of the Put-Call Parity formula which include the effects of dividends:

\[ P_e + S e^{-\delta T} = C_e + K e^{-rT} \]  \hspace{1cm} (continuous)

\[ P_e + S(0) - \delta \sum_{i=1}^{n} S(t_i^-) e^{-r_t i} = C_e + K e^{-rT} \]  \hspace{1cm} (discrete)

We do not have pricing formulas for the options themselves. We explore modifications and extensions to the Black-Scholes partial differential equation and its solution in this lesson.
The non-dividend-paying stock is assumed to obey the stochastic process

\[ dS = \mu S \, dt + \sigma S \, dW(t) \]

and the European call solves the initial boundary value problem:

\[
\begin{align*}
    rF &= F_t + rSF_S + \frac{1}{2}\sigma^2 S^2 F_{SS} \quad \text{for } (S, t) \text{ in } [0, \infty) \times [0, T],
    \\
    F(S, T) &= (S(T) - K)^+ \quad \text{for } S > 0,
    \\
    F(0, t) &= 0 \quad \text{for } 0 \leq t < T,
    \\
    F(S, t) &= S - Ke^{-r(T-t)} \quad \text{as } S \to \infty.
\end{align*}
\]
Assumption: the stock pays dividends at a continuous rate proportional to the value of the stock

- What is a suitable expression for the dividend yield (dividend paid per unit time)?

- How much dividend is paid in a short time interval $dt$?

- What stochastic differential equation would the value of the stock paying a continuous proportional dividend obey?
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  \[
  \text{dividend paid} = \delta S \, dt
  \]

- What stochastic differential equation would the value of the stock paying a continuous proportional dividend obey?

\[
dS = (\mu - \delta) S \, dt + \sigma S \, dW(t)
\]
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- What stochastic differential equation would the value of the stock paying a continuous proportional dividend obey?

  \[
  dS = (\mu - \delta) S \, dt + \sigma S \, dW(t)
  \]
Suppose $F(S, t)$ is the value of a European call option on the stock paying a continuous dividend, $F$ obeys the following stochastic differential equation:

$$dF = \left( (\mu - \delta)SF_S + \frac{1}{2}\sigma^2 S^2 F_{SS} + F_t \right) dt + \sigma SF_S dW(t).$$
Suppose $F(S, t)$ is the value of a European call option on the stock paying a continuous dividend, $F$ obeys the following stochastic differential equation:

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As before, we wish to eliminate the random part of this equation by creating a portfolio of a long position in the call option and a short position in $\Delta$ shares of the stock.

$$P = F - (\Delta)S$$
One share of stock pays $\delta S \, dt$ in dividends during a time interval of length $dt$, thus $\Delta$ shares of stock pays

$$\delta(\Delta) S \, dt$$

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The portfolio changes in value

$$dP = d(F - (\Delta)S) - \delta(\Delta)S \, dt$$
$$= dF - (\Delta)dS - \delta(\Delta)S \, dt$$
$$= \left( (\mu - \delta)SF_S + \frac{1}{2}\sigma^2 S^2 F_{SS} + F_t \right) dt + \sigma SF_S \, dW(t)$$
$$- (\Delta) \left( (\mu - \delta)S \, dt + \sigma S \, dW(t) \right) - \delta(\Delta)S \, dt$$
$$= \left( (\mu - \delta)S(F_S - \Delta) + \frac{1}{2}\sigma^2 S^2 F_{SS} + F_t - \delta(\Delta)S \right) dt$$
$$+ \sigma S(F_S - \Delta) \, dW(t).$$
Choose $\Delta = F_S$ and the portfolio obeys the stochastic differential equation:

$$dP = \left(\frac{1}{2} \sigma^2 S^2 F_{SS} + F_t - \delta S F_S\right) dt.$$ 

In the absence of arbitrage the change in the value of the portfolio should be the same as the interest earned by an equivalent amount of cash.

$$dP = r(F - (\Delta)S) \, dt$$
Choose $\Delta = F_S$ and the portfolio obeys the stochastic differential equation:

$$dP = \left( \frac{1}{2} \sigma^2 S^2 F_{SS} + F_t - \delta S F_S \right) dt.$$ 

In the absence of arbitrage the change in the value of the portfolio should be the same as the interest earned by an equivalent amount of cash.

$$dP = r(F - (\Delta)S) dt$$

Thus the Black-Scholes partial differential equation for the stock paying continuous dividends becomes

$$rF = F_t + \frac{1}{2} \sigma^2 S^2 F_{SS} + (r - \delta) S F_S.$$
Payoff of the call option at expiry: \( F(S, T) = (S(T) - K)^+ \).

Boundary condition at \( S = 0 \) is \( F(0, t) = 0 \).

Boundary condition as \( S \to \infty \):

\[
F(S, t) = P_e + Se^{-\delta(T-t)} - Ke^{-r(T-t)}
\]

\[
\lim_{S \to \infty} F(S, t) = \lim_{S \to \infty} \left( P_e + Se^{-\delta(T-t)} - Ke^{-r(T-t)} \right) = Se^{-\delta(T-t)} - Ke^{-r(T-t)}.
\]
Define the function $G(S, t) = e^{\delta(T-t)}F(S, t)$, then

\[
G(S, T) = e^{\delta(T-T)}F(S, T) = (S(T) - K)^+
\]

\[
G(0, t) = e^{\delta(T-t)}F(0, t) = 0
\]

\[
\lim_{S \to \infty} G(S, t) = e^{\delta(T-t)} \left( Se^{-\delta(T-t)} - Ke^{-r(T-t)} \right)
\]

\[
= S - e^{-(r-\delta)(T-t)}
\]

\[
F_S = e^{-\delta(T-t)} G_S
\]

\[
F_{SS} = e^{-\delta(T-t)} G_{SS}
\]

\[
F_t = e^{-\delta(T-t)} (\delta G + G_t).
\]

Substitute these expressions into the partial differential equation, boundary conditions, and the final condition for the European call option on the stock paying continuous dividends.
(r − δ)G = G_t + \frac{1}{2}\sigma^2 S^2 G_{SS} + (r − δ)SG_S

G(S, T) = (S(T) − K)^+

G(0, t) = 0

\lim_{S \to \infty} G(S, t) = S − e^{−(r−δ)(T−t)}

**Remark:** this is exactly the same initial boundary value problem we have already solved except \( r \) has been replaced by \( r − \delta \).
For a stock paying a continuous, proportional dividend at rate $\delta$ the value of a European option is given by the formulas

$$w = \frac{\ln(S/K) + (r - \delta + \sigma^2/2)(T - t)}{\sigma \sqrt{T - t}}$$

$$C^\delta_e = e^{-\delta(T-t)}S\Phi(w) - Ke^{-r(T-t)}\Phi\left(w - \sigma \sqrt{T - t}\right)$$

$$P^\delta_e = Ke^{-r(T-t)}\Phi\left(\sigma \sqrt{T - t} - w\right) - e^{-\delta(T-t)}S\Phi(-w)$$
The lighter curve represents a call option on a stock paying no dividends, while the heavier curve represents an otherwise identical call option paying a continuous dividend.
Suppose the current price of a security is $62 per share. The continuously compounded interest rate is 10% per year. The volatility of the price of the security is $\sigma = 20\%$ per year. The stock pays dividends continuously at a rate of $\delta = 3\%$ per year. Find the cost of a five-month European call option with a strike price of $60$ per share.
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\[
T = \frac{5}{12}, \quad t = 0, \quad r = 0.10, \quad \sigma = 0.20, \\
S = 62, \quad K = 60, \quad \delta = 0.03
\]

Using the formula for $w$ and $C_\delta^e$ we have

\[
w \approx 0.544463 \\
C_\delta^e \approx 5.24
\]
Example: Call Option

Suppose the current price of a security is $62 per share. The continuously compounded interest rate is 10% per year. The volatility of the price of the security is $\sigma = 20\%$ per year. The stock pays dividends continuously at a rate of $\delta = 3\%$ per year. Find the cost of a five-month European call option with a strike price of $60$ per share.

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\[ S = 62, \quad K = 60, \quad \delta = 0.03 \]

Using the formula for $w$ and $C_\delta$ we have

\[ w \approx 0.544463 \]
\[ C_\delta \approx 5.24 \]

Without the dividend we calculate $C_e \approx 5.80$. 
Example: Put Option

Suppose the current price of a security is $97 per share. The stock pays a continuous dividend at a yield of 6.5% per year. The continuously compounded interest rate is 8% per year. The volatility of the price of the security is $\sigma = 45\%$ per year. Find the cost of a three-month European put option with a strike price of $95 per share.
Suppose the current price of a security is $97 per share. The stock pays a continuous dividend at a yield of 6.5% per year. The continuously compounded interest rate is 8% per year. The volatility of the price of the security is $\sigma = 45\%$ per year. Find the cost of a three-month European put option with a strike price of $95 per share.

$$T = \frac{1}{4}, \quad t = 0, \quad r = 0.08, \quad \sigma = 0.45,$$

$$\delta = 0.065, \quad S = 97, \quad K = 95.$$

Using the formulas for $w$ and $P^\delta_e$ we obtain

$$w \approx 0.221763$$

$$P^\delta_e \approx 7.34$$
The rate of change in the price of a European call option on a stock paying continuous dividends is

$$\rho_C^\delta = \frac{\partial C^\delta}{\partial \delta} = -S(T - t)e^{-\delta(T-t)}\Phi(w).$$

For a European put option

$$\rho_P^\delta = \frac{\partial P^\delta}{\partial \delta} = S(T - t)e^{-\delta(T-t)}(1 - \Phi(w)).$$
The rate of change in the price of a European call option on a stock paying continuous dividends is

\[ \rho_C = \frac{\partial C_\delta}{\partial \delta} = -S(T - t)e^{-\delta(T-t)}\Phi(w). \]

For a European put option

\[ \rho_P = \frac{\partial P_\delta}{\partial \delta} = S(T - t)e^{-\delta(T-t)}(1 - \Phi(w)). \]

**Remark:** some authors call this Greek, Psi, and denote it \(\Psi\).
Dividends Influence on Other Greeks

The presence of the continuous dividend rate $\delta$, in the Call and Put formulas alters the previously discussed Greeks.

$$w = \ln\left(\frac{S}{K}\right) + (r - \delta + \sigma^2/2)(T - t) \frac{1}{\sigma \sqrt{T - t}}$$

$$C^\delta_e = e^{-\delta(T-t)} S \Phi (w) - Ke^{-r(T-t)} \Phi \left( w - \sigma \sqrt{T - t} \right)$$

$$P^\delta_e = Ke^{-r(T-t)} \Phi \left( w - \sigma \sqrt{T - t} \right) - e^{-\delta(T-t)} S \Phi (-w)$$

Find Delta, Gamma, Rho, Theta, and Vega.
For a Call:

\[
\frac{\partial C^\delta_e}{\partial S} = e^{-\delta(T-t)} \Phi (w) + e^{-\delta(T-t)} \phi (w) \frac{\partial w}{\partial S} \\
- Ke^{-r(T-t)} \phi \left( w - \sigma \sqrt{T-t} \right) \frac{\partial w}{\partial S} \\
\]

\[= e^{-\delta(T-t)} \Phi (w).\]
For a Call:

\[
\frac{\partial C_\delta}{\partial S} = e^{-\delta(T-t)} \Phi (w) + e^{-\delta(T-t)} \phi (w) \frac{\partial w}{\partial S} - Ke^{-r(T-t)} \phi \left( w - \sigma \sqrt{T-t} \right) \frac{\partial w}{\partial S} = e^{-\delta(T-t)} \Phi (w) .
\]

For a Put:

\[
\frac{\partial P_\delta}{\partial S} = Ke^{-r(T-t)} \Phi \left( w - \sigma \sqrt{T-t} \right) \frac{\partial w}{\partial S} - e^{-\delta(T-t)} \Phi (-w) + e^{-\delta(T-t)} S \phi (-w) \frac{\partial w}{\partial S} = -e^{-\delta(T-t)} \Phi (-w) .
\]
For a Call:

\[
\frac{\partial^2 C_e^\delta}{\partial S^2} = e^{-\delta(T-t)} \phi(w) \frac{\partial w}{\partial S}
\]
\[
= e^{-\delta(T-t)} \frac{\phi(w)}{\sigma S \sqrt{T-t}}.
\]
For a Call:

\[
\frac{\partial^2 C_e}{\partial S^2} = e^{-\delta(T-t)} \phi (w) \frac{\partial w}{\partial S} \\
= e^{-\delta(T-t)} \frac{\phi (w)}{\sigma S \sqrt{T-t}}.
\]

For a Put:

\[
\frac{\partial^2 P_e}{\partial S^2} = e^{-\delta(T-t)} \phi (-w) \frac{\partial w}{\partial S} \\
= e^{-\delta(T-t)} \frac{\phi (w)}{\sigma S \sqrt{T-t}}.
\]
For a Call:

\[
\frac{\partial C^\delta}{\partial r} = e^{-\delta(T-t)} S \phi(w) \frac{\partial w}{\partial r} + K(T - t)e^{-r(T-t)} \Phi \left( w - \sigma \sqrt{T-t} \right) \\
- Ke^{-r(T-t)} \phi \left( w - \sigma \sqrt{T-t} \right) \frac{\partial w}{\partial r} \\
= K(T - t)e^{-r(T-t)} \Phi \left( w - \sigma \sqrt{T-t} \right).
\]
Rho

For a Call:

\[
\frac{\partial C_e^\delta}{\partial r} = e^{-\delta(T-t)} S \phi (w) \frac{\partial w}{\partial r} + K(T-t)e^{-r(T-t)} \Phi \left( w - \sigma \sqrt{T-t} \right) \\
- Ke^{-r(T-t)} \phi \left( w - \sigma \sqrt{T-t} \right) \frac{\partial w}{\partial r} \\
= K(T-t)e^{-r(T-t)} \Phi \left( w - \sigma \sqrt{T-t} \right).
\]

For a Put:

\[
\frac{\partial P_e^\delta}{\partial r} = -K(T-t)e^{-r(T-t)} \Phi \left( w - \sigma \sqrt{T-t} \right) \\
+ Ke^{-r(T-t)} \phi \left( w - \sigma \sqrt{T-t} \right) \frac{\partial w}{\partial r} + e^{-\delta(T-t)} S \phi (-w) \frac{\partial w}{\partial r} \\
= -K(T-t)e^{-r(T-t)} \Phi \left( \sigma \sqrt{T-t} - w \right).
\]
For a Call:

$$\frac{\partial C^\delta_e}{\partial t} = \delta e^{-\delta(T-t)} S \Phi (w) + e^{-\delta(T-t)} S \phi (w) \frac{\partial w}{\partial t}$$

$$- Ke^{-r(T-t)} \Phi (w - \sigma \sqrt{T - t})$$

$$- Ke^{-r(T-t)} \phi (w - \sigma \sqrt{T - t}) \left( \frac{\partial w}{\partial t} + \frac{\sigma}{2 \sqrt{T - t}} \right)$$

$$= \delta Se^{-\delta(T-t)} \Phi (w) - Ke^{-r(T-t)} \Phi (w - \sigma \sqrt{T - t})$$

$$- e^{-\delta(T-t)} \frac{\sigma S \phi (w)}{2 \sqrt{T - t}}.$$
For a Put:

\[
\frac{\partial P_e^{\delta}}{\partial t} = K r e^{-r(T-t)} \Phi \left( w - \sigma \sqrt{T-t} \right) \\
+ Ke^{-r(T-t)} \phi \left( w - \sigma \sqrt{T-t} \right) \left( \frac{\partial w}{\partial t} + \frac{\sigma}{2\sqrt{T-t}} \right) \\
- \delta e^{-\delta(T-t)} S \Phi (-w) + e^{-\delta(T-t)} S \phi (w) \frac{\partial w}{\partial t} \\
= K r e^{-r(T-t)} \Phi \left( \sigma \sqrt{T-t} - w \right) - \delta Se^{-\delta(T-t)} \Phi (-w) \\
- e^{-\delta(T-t)} \frac{\sigma S \phi (w)}{2\sqrt{T-t}}.
\]
For a Call:

\[
\frac{\partial C^\delta_e}{\partial \sigma} = e^{-\delta(T-t)} S \phi (w) \frac{\partial w}{\partial \sigma}
- Ke^{-r(T-t)} \phi \left( w - \sigma \sqrt{T-t} \right) \left( \frac{\partial w}{\partial \sigma} - \sqrt{T-t} \right)
= Se^{-\delta(T-t)} \sqrt{T-t} \phi (w).
\]
For a Call:

\[
\frac{\partial C^\delta}{\partial \sigma} = e^{-\delta(T-t)} S\phi(w) \frac{\partial w}{\partial \sigma} \\
- Ke^{-r(T-t)} \phi \left( w - \sigma \sqrt{T-t} \right) \left( \frac{\partial w}{\partial \sigma} - \sqrt{T-t} \right) \\
= Se^{-\delta(T-t)} \sqrt{T-t} \phi \left( w - \sigma \sqrt{T-t} \right).
\]

For a Put:

\[
\frac{\partial P^\delta}{\partial \sigma} = Ke^{-r(T-t)} \phi \left( w - \sigma \sqrt{T-t} \right) \left( \frac{\partial w}{\partial \sigma} - \sqrt{T-t} \right) \\
+ e^{-\delta(T-t)} S\phi(-w) \frac{\partial w}{\partial \sigma} \\
= Ke^{-r(T-t)} \sqrt{T-t} \phi \left( w - \sigma \sqrt{T-t} \right).
\]
We have argued through absence of arbitrage that a stock must decrease in value by the amount of any dividend paid.

If a stock will pay a proportional dividend $d_y S(t)$ at time $t_d$ then

$$S(t_d^+) = (1 - d_y) S(t_d^-)$$

where

$$S(t_d^-) = \lim_{t \to t_d^-} S(t) \quad \text{and} \quad S(t_d^+) = \lim_{t \to t_d^+} S(t).$$
The discontinuous jump in the price of a stock across a dividend date.
The **Dirac Delta function** \( D(t) \), is a function with the following properties:

- \( D(t) = 0 \) for all \( t \neq 0 \).
- \( \int_{-\infty}^{\infty} D(t) \, dt = 1 \)
- \( \int_{-\infty}^{\infty} f(t)D(t) \, dt = f(0) \) for any continuous function \( f(t) \) defined on the real numbers.

Thus if \( S(t) \) represents a stock which pays a single discrete dividend at time \( t_d \) then \( S \) obeys the stochastic differential equation:

\[
dS = (\mu - dY)D(t - td)S \, dt + \sigma S \, dW(t)
\]
The Dirac Delta function $D(t)$, is a function with the following properties:

- $D(t) = 0$ for all $t \neq 0$.
- $\int_{-\infty}^{\infty} D(t) \, dt = 1$
- $\int_{-\infty}^{\infty} f(t)D(t) \, dt = f(0)$ for any continuous function $f(t)$ defined on the real numbers.

Thus if $S(t)$ represents a stock which pays a single discrete dividend at time $t_d$ then $S$ obeys the stochastic differential equation:

$$dS = (\mu - d_y D(t - t_d))S \, dt + \sigma S \, dW(t).$$
Solving for $S(t)$

If $dS = (\mu - d_y D(t - t_d)) S \, dt + \sigma S \, dW(t)$ and $Y = \ln S$ then by Itô’s Lemma

$$dY = \left( \mu - d_y D(t - t_d) - \frac{1}{2} \sigma^2 \right) dt + \sigma \, dW(t)$$

$$Y(t) = Y(0) \begin{cases} 
(\mu - \frac{1}{2} \sigma^2) \, t + \sigma \, W(t) & \text{if } t < t_d, \\
(\mu - \frac{1}{2} \sigma^2) \, t - d_y + \sigma \, W(t) & \text{if } t \geq t_d.
\end{cases}$$

$$S(t) = S(0) \begin{cases} 
e^{(\mu-\sigma^2/2)t+\sigma W(t)} & \text{if } t < t_d, \\
e^{(\mu-\sigma^2/2)t-d_y+\sigma W(t)} & \text{if } t \geq t_d.
\end{cases}$$
\[ S(t_d^-) - S(t_d^+) = S(0)e^{(\mu-\sigma^2/2)t_d+\sigma W(t_d)} - S(0)e^{(\mu-\sigma^2/2)t_d-dy+\sigma W(t_d)} \]
\[ = S(0)e^{(\mu-\sigma^2/2)t_d+\sigma W(t_d)} \left[ 1 - e^{-dy} \right] \]
\[ = S(t_d^-) \left( 1 - e^{-dy} \right) \]
\[ S(t_d^+) = S(t_d^-)e^{-dy} \]
Since $S(t)$ is lognormal,

$$
E[S(t)] = \begin{cases} 
S(0)e^{\mu t} & \text{if } t < t_d, \\
S(0)e^{\mu t - d_y} & \text{if } t \geq t_d.
\end{cases}
$$
In the absence of arbitrage, the price of the option must be continuous across the dividend date.

\[
\lim_{t \to t_d^-} C^e(S(t), t) = \lim_{t \to t_d^+} C^e(S(t), t)
\]

\[
C^e(S(t_d^-), t_d^-) = C^e(S(t_d^+), t_d^+)
\]

\[
C^e(S(t_d^-), t_d^-) = C^e(S(t_d^-)e^{-dy}, t_d^+)
\]
Continuity of Option Price (2 of 2)

\[ C^e(S(t_d^-), t_d^-) = C^e(S(t_d^-)e^{-dy}, t_d^+) \]

Remarks:

- The value of the call option will change discontinuously across the dividend date as a function of \( S \).
- However, the price of the option is made continuous by equating the value of the option just before the dividend is paid with the value of the option just after the dividend is paid.
- The value of the stock underlying the option has been adjusted to \( S(t_d^+) = S(t_d^-)e^{-dy} \).
Separate the life of the option into two intervals, \([0, t_d)\) and \((t_d, T]\).

- On the interval \((t_d, T]\) no dividends are paid and the original formula for the price of a call option can be used.
- The post dividend value of the stock is \(S e^{-d_y}\).
- For \(t_d < t \leq T\),

\[
C_\delta^\delta = e^{-d_y} S \Phi (w) - K \Phi \left( w - \sigma \sqrt{T-t} \right)
\]
At $t = t_d^+$ the value of the call option is

$$C^e(S(t_d^+, t_d^+), t_d^+) = S(t_d^+) \Phi (w) - Ke^{-r(T-t_d^+)} \Phi \left( w - \sigma \sqrt{T-t_d^+} \right).$$

Immediately before the dividend date the value of the call option is

$$C^e(S(t_d^-, t_d^-), t_d^-) = C^e(S(t_d^-) e^{-d_y}, t_d^+) = S(t_d^-) e^{-d_y} \Phi (w) - Ke^{-r(T-t_d^+)} \Phi \left( w - \sigma \sqrt{T-t_d^+} \right).$$

**Note:** the price of the stock underlying the option has been scaled by the factor $e^{-d_y}$. 
Define $\hat{S} = Se^{-dy}$ then

\[
\frac{\partial}{\partial S} [F(S, t)] = e^{-dy} \frac{\partial}{\partial \hat{S}} [F(S, t)] \quad \text{and} \\
\frac{\partial^2}{\partial S^2} [F(S, t)] = e^{-2dy} \frac{\partial^2}{\partial \hat{S}^2} [F(S, t)].
\]
Define $\hat{S} = Se^{-dy}$ then

$$\frac{\partial}{\partial S} [F(S, t)] = e^{-dy} \frac{\partial}{\partial \hat{S}} [F(S, t)] \quad \text{and}$$

$$\frac{\partial^2}{\partial S^2} [F(S, t)] = e^{-2dy} \frac{\partial^2}{\partial \hat{S}^2} [F(S, t)].$$

Substituting these into the Black-Scholes partial differential equation yields

$$rF = Ft + \frac{1}{2} \sigma^2 \hat{S}^2 F_{\hat{S}\hat{S}} + r\hat{S}F_{\hat{S}}.$$
We have the PDE:

\[ rF = F_t + \frac{1}{2} \sigma^2 \hat{S}^2 F_{\hat{S}\hat{S}} + r\hat{S}F_{\hat{S}}. \]

What about the boundary and final conditions?
We have the PDE:

\[ rF = F_t + \frac{1}{2} \sigma^2 \hat{S}^2 F_{\hat{S}\hat{S}} + r\hat{S}F_{\hat{S}}. \]

What about the boundary and final conditions?

\[ F(\hat{S}, T) = (Se^{-d_y} - K)^+ = e^{-d_y} (S - Ke^{d_y})^+ \]
\[ F(0, t) = F(0 \cdot e^{-d_y}, t) = 0 \]
\[ \lim_{\hat{S} \to \infty} F(\hat{S}, t) = e^{-d_y} \left( S - Ke^{d_y-r(T-t)} \right) \]
For $0 \leq t < t_d$ use the established European call option pricing formula for $e^{-d_y}$ call options with a strike price of $Ke^{d_y}$.

Thus for $t < t_d$,

$$C_e^\delta = e^{-d_y} \left[ S\Phi (w) - Ke^{d_y-r(T-t)}\Phi \left( w - \sigma \sqrt{T-t} \right) \right].$$
The value of a European call option on a stock paying a single discrete, proportional dividend at \( t = t_d \). The value of the stock is constant where \( S(t) = K \), the strike price.
The value of a European call option immediately before (bold) and after a discrete dividend payment. Note that while the value of the stock instantaneously decreases in value by 10% as the dividend is paid, the value of the call option is continuous.
The value of a European call option written on a stock paying a single discrete, proportional dividend during the life of the option can be written as a piecewise-defined function.

\[
C^{e,\delta}(S, t) = \begin{cases} 
  e^{-d_y} \left[ S \Phi (w) - K e^{d_y} \Phi (w - \sigma \sqrt{T - t}) \right] & \text{if } t < t_d, \\
  e^{-d_y} S \Phi (w) - K \Phi (w - \sigma \sqrt{T - t}) & \text{if } t \geq t_d.
\end{cases}
\]

**Note:**
- In the pre-dividend portion of this formula \( K e^{d_y} \) is used for the strike price when calculating \( w \) and \( C_e \).
- In the post-dividend formula \( S e^{-d_y} \) is used for the value of the underlying stock.
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author: J. Robert Buchanan


address: 27 Warren St., Suite 401–402, Hackensack, NJ 07601

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