The Greeks: Derivatives of Option Prices
An Undergraduate Introduction to Financial Mathematics

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Sensitivity Analysis

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In the world a finance an option is an example of a **derivative**, any financial instrument whose value is *derived* from that of an underlying security.

Here we will calculate the partial derivatives (in the sense of calculus) of option value formulas. These partial derivatives will allow us to determine how sensitive the values of options are to changes in independent variables and parameters.
Black-Scholes Option Pricing Formulas

$$w = \frac{\ln(S/K) + (r + \sigma^2/2)(T - t)}{\sigma \sqrt{T - t}}$$

$$C(S, t) = S\phi(w) - Ke^{-r(T-t)}\phi(w - \sigma\sqrt{T - t})$$

$$P(S, t) = Ke^{-r(T-t)}\phi(\sigma\sqrt{T - t} - w) - S\phi(-w)$$
The function $\phi(w)$ is the cumulative distribution function

$$\phi(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{w} e^{-x^2/2} \, dx$$
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$$\phi(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{w} e^{-x^2/2} dx$$

which by the Fundamental Theorem of Calculus has derivative

$$\phi'(w) = \frac{1}{\sqrt{2\pi}} e^{-w^2/2}.$$
Partial Derivatives of $w$

$$w = \ln(\frac{S}{K}) + (r + \frac{\sigma^2}{2})(T - t)$$
Partial Derivatives of $w$

\[
\begin{align*}
    w &= \frac{\ln(S/K) + (r + \sigma^2/2)(T - t)}{\sigma \sqrt{T - t}} \\
    \frac{\partial w}{\partial t} &= \frac{1}{2\sigma \sqrt{T - t}} \left( \frac{\ln(S/K)}{T - t} - r - \frac{\sigma^2}{2} \right)
\end{align*}
\]
Partial Derivatives of $w$

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\begin{align*}
w &= \frac{\ln(S/K) + (r + \sigma^2/2)(T - t)}{\sigma \sqrt{T - t}} \\
\frac{\partial w}{\partial t} &= \frac{1}{2\sigma \sqrt{T - t}} \left( \frac{\ln(S/K)}{T - t} - r - \frac{\sigma^2}{2} \right) \\
\frac{\partial w}{\partial S} &= \frac{1}{\sigma S \sqrt{T - t}}
\end{align*}
\]
Partial Derivatives of $w$

\[ w = \frac{\ln(S/K) + (r + \sigma^2/2)(T - t)}{\sigma \sqrt{T - t}} \]

\[ \frac{\partial w}{\partial t} = \frac{1}{2\sigma \sqrt{T - t}} \left( \frac{\ln(S/K)}{T - t} - r - \frac{\sigma^2}{2} \right) \]

\[ \frac{\partial w}{\partial S} = \frac{1}{\sigma S \sqrt{T - t}} \]

\[ \frac{\partial w}{\partial r} = \frac{\sqrt{T - t}}{\sigma} \]
Partial Derivatives of $w$

$$w = \frac{\ln(S/K) + (r + \sigma^2/2)(T - t)}{\sigma \sqrt{T - t}}$$

$$\frac{\partial w}{\partial t} = \frac{1}{2\sigma \sqrt{T - t}} \left( \frac{\ln(S/K)}{T - t} - r - \frac{\sigma^2}{2} \right)$$

$$\frac{\partial w}{\partial S} = \frac{1}{\sigma S \sqrt{T - t}}$$

$$\frac{\partial w}{\partial r} = \frac{\sqrt{T - t}}{\sigma}$$

$$\frac{\partial w}{\partial \sigma} = \frac{\sqrt{T - t} - w}{\sigma}$$
Theta $\Theta$ is the partial derivative with respect to time $t$.

Time is the only independent variable we are certain will change before expiry. It is also the only deterministic independent variable.
\[
C = S \phi(w) - Ke^{-r(T-t)} \phi(w - \sigma \sqrt{T-t}) \\
\frac{\partial C}{\partial t} = S \phi'(w) \frac{\partial w}{\partial t} - rKe^{-r(T-t)} \phi(w - \sigma \sqrt{T-t}) \\
- Ke^{-r(T-t)} \phi'(w - \sigma \sqrt{T-t}) \left[ \frac{\partial w}{\partial t} + \frac{\sigma}{2\sqrt{T-t}} \right]
\]
\[
C = S\phi(w) - Ke^{-r(T-t)}\phi(w - \sigma\sqrt{T-t})
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\frac{\partial C}{\partial t} = S\phi'(w)\frac{\partial w}{\partial t} - rKe^{-r(T-t)}\phi(w - \sigma\sqrt{T-t})
\]

\[
-Ke^{-r(T-t)}\phi'(w - \sigma\sqrt{T-t})\left[\frac{\partial w}{\partial t} + \frac{\sigma}{2\sqrt{T-t}}\right]
\]

\[
= Se^{-w^2/2} - Ke^{-r(T-t)-(w-\sigma\sqrt{T-t})^2/2}
\]

\[
\frac{2\sigma}{2\sqrt{2\pi(T-t)}}\left(\frac{\ln(S/K)}{T-t} - r - \sigma^2/2\right)
\]

\[
-Ke^{-r(T-t)}\left[r\phi(w - \sigma\sqrt{T-t}) + \frac{\sigma e^{-(w-\sigma\sqrt{T-t})^2/2}}{2\sqrt{2\pi(T-t)}}\right]
\]
For a European Put

\[
\frac{\partial P}{\partial t} = \frac{Se^{-w^2/2}}{2\sigma\sqrt{2\pi(T-t)}} \left( \frac{\ln(S/K)}{T-t} - r - \frac{\sigma^2}{2} \right) \\
+ Ke^{-r(T-t)} \phi(\sigma\sqrt{T-t} - w) \\
- Ke^{-r(T-t)-(w-\sigma\sqrt{T-t})^2/2} \left( \frac{\ln(S/K)}{T-t} - r + \frac{\sigma^2}{2} \right)
\]
\( \Delta \) was involved in the derivation of the Black-Scholes PDE and is defined to be the partial derivative with respect to the price of the security.

\[
\begin{align*}
  C &= S \phi(w) - Ke^{-r(T-t)} \phi(w - \sigma \sqrt{T-t}) \\
  \frac{\partial C}{\partial S} &= \phi(w) + \left( S \phi'(w) - Ke^{-r(T-t)} \phi'(w - \sigma \sqrt{T-t}) \right) \frac{\partial w}{\partial S}
\end{align*}
\]
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&= \phi(w)
\end{align*}
\]
$\Delta$ was involved in the derivation of the Black-Scholes PDE and is defined to be the partial derivative with respect to the price of the security.

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C = S\phi(w) - Ke^{-r(T-t)}\phi(w - \sigma\sqrt{T-t}) \\
\frac{\partial C}{\partial S} = \phi(w) + \left( S\phi'(w) - Ke^{-r(T-t)}\phi'(w - \sigma\sqrt{T-t}) \right) \frac{\partial w}{\partial S} \\
= \phi(w)
\]

**Question:** what is the range of Delta for a European call option?
Recall the Put-Call Parity formula:

\[ P + S = C + Ke^{-r(T-t)} \]
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\[ P + S = C + Ke^{-r(T-t)} \]

\[ \frac{\partial}{\partial S}(P + S) = \frac{\partial}{\partial S}(C + Ke^{-r(T-t)}) \]
Recall the Put-Call Parity formula:

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P + S = C + Ke^{-r(T-t)}
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\frac{\partial}{\partial S}(P + S) = \frac{\partial}{\partial S}(C + Ke^{-r(T-t)})
\]

\[
\frac{\partial P}{\partial S} + 1 = \frac{\partial C}{\partial S}
\]

\[
\frac{\partial P}{\partial S} = \phi(w) - 1
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Recall the Put-Call Parity formula:

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\frac{\partial P}{\partial S} + 1 = \frac{\partial C}{\partial S}
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\[
\frac{\partial P}{\partial S} = \phi(w) - 1
\]

**Question:** what is the range of Delta for a European put option?
The current price of a stock is $77 and its volatility is 35% per year. The risk-free interest rate is 3.25% per year. A portfolio is constructed consisting of one six-month European call option with a strike price of $80 and the cash obtained from shorting $\Delta$ shares of the stock. The portfolio’s value is non-random. What is $\Delta$?
The assumption the portfolio’s value is non-random is the assumption

\[ (\Delta) S - C = \left( \frac{\partial C}{\partial S} \right) S - C = 0 \]

made in deriving the Black-Scholes equation.
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\[(\Delta) S - C = \left( \frac{\partial C}{\partial S} \right) S - C = 0\]

made in deriving the Black-Scholes equation.

\[
S = 77 \quad \sigma = 0.35 \quad T = \frac{6}{12} \\
r = 0.0325 \quad K = 80 \quad t = 0
\]

Using these values

\[
w = \ln\left( \frac{S}{K} \right) + \left( r + \frac{\sigma^2}{2} \right) (T - t) \\
\frac{\partial C}{\partial S} = \Delta = \phi(w)
\]
The assumption the portfolio’s value is non-random is the assumption

$$(\Delta)S - C = \left(\frac{\partial C}{\partial S}\right)S - C = 0$$

made in deriving the Black-Scholes equation.

$\begin{align*}
S &= 77 \\
\sigma &= 0.35 \\
T &= \frac{6}{12} \\
r &= 0.0325 \\
K &= 80 \\
t &= 0
\end{align*}$

Using these values

$$w = \frac{\ln(S/K) + (r + \sigma^2/2)(T - t)}{\sigma \sqrt{T - t}} \approx 0.0349666$$

$$\frac{\partial C}{\partial S} = \Delta = \phi(w)$$
The assumption the portfolio’s value is non-random is the assumption

$$(\Delta)S - C = \left(\frac{\partial C}{\partial S}\right)S - C = 0$$

made in deriving the Black-Scholes equation.

Using these values

$$S = 77, \quad \sigma = 0.35, \quad T = \frac{6}{12}, \quad r = 0.0325, \quad K = 80, \quad t = 0$$

$$w = \ln(S/K) + (r + \sigma^2/2)(T - t) \approx 0.0349666$$

$$\frac{\partial C}{\partial S} = \Delta = \phi(w) \approx 0.513947$$
Gamma is the second partial derivative with respect to $S$, thus

$$
\Gamma = \frac{\partial}{\partial S} \phi(w)
\quad = \quad \phi'(w) \frac{\partial w}{\partial S}
\quad = \quad \phi'(w) \frac{\partial w}{\partial S}
\quad = \quad \frac{e^{-w^2/2}}{\sigma S \sqrt{2\pi(T-t)}}.
$$
Gamma vs. $S(0)$

$K = 100, \quad \sigma = 0.25 \quad T = 1 \quad r = 0.0325$
\[ S(0) = 100, \quad \sigma = 0.25 \quad r = 0.0325 \]
Remember the Black-Scholes PDE:

\[ rF = F_t + rSF_S + \frac{1}{2} \sigma^2 S^2 F_{SS} \]
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Since \( F_t = \Theta \), \( \Delta = F_S \), and \( F_{SS} = \Gamma \) then the Black-Scholes equation can be thought of as

\[ rF = \Theta + rS\Delta + \frac{1}{2} \sigma^2 S^2 \Gamma. \]
Vega $\nu$ (1 of 2)

Vega is the partial derivative with respect to volatility $\sigma$.

\[
C = S\phi(w) - Ke^{-r(T-t)}\phi(w - \sigma\sqrt{T-t})
\]

\[
\frac{\partial C}{\partial \sigma} = S\phi'(w)\frac{\partial w}{\partial \sigma} - Ke^{-r(T-t)}\phi'(w - \sigma\sqrt{T-t}) \left( \frac{\partial w}{\partial \sigma} - \sqrt{T-t} \right)
\]

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\]
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\frac{\partial C}{\partial \sigma} = S\phi'(w)\frac{\partial w}{\partial \sigma} - Ke^{-r(T-t)}\phi'(w - \sigma\sqrt{T-t})\left(\frac{\partial w}{\partial \sigma} - \sqrt{T-t}\right)
\]
\[
= \frac{S\sqrt{T-t}}{\sqrt{2\pi}}e^{-w^2/2}
\]
According to the Put-Call Parity formula:

\[ P + S = C + Ke^{-r(T-t)} \]
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\[ \frac{\partial}{\partial \sigma}(P + S) = \frac{\partial}{\partial \sigma}(C + Ke^{-r(T-t)}) \]
According to the Put-Call Parity formula:

\[ P + S = C + Ke^{-r(T-t)} \]

\[ \frac{\partial}{\partial \sigma} (P + S) = \frac{\partial}{\partial \sigma} (C + Ke^{-r(T-t)}) \]

\[ \frac{\partial P}{\partial \sigma} = \frac{\partial C}{\partial \sigma} \]

\[ \frac{\partial P}{\partial \sigma} = \frac{S\sqrt{T-t}}{\sqrt{2\pi}} e^{-w^2/2} \]

**Remark:** vega is identical for puts and calls.
Example

Consider a three-month European put option on a stock whose current value is $200 and whose volatility is 30%. The option has a strike price of $195 and the risk-free interest rate is 6.25%.

1. Find the vega of the option.
2. If the volatility of the stock increases to 31%, approximate the change in the value of the put.
\begin{align*}
S &= 200 \quad \sigma = 0.30 \quad T = 3/12 \\
K &= 195 \quad r = 0.0625 \quad t = 0
\end{align*}
\[ S = 200 \quad \sigma = 0.30 \quad T = 3/12 \]
\[ K = 195 \quad r = 0.0625 \quad t = 0 \]

Using these values

\[ w = \frac{\ln(S/K) + (r + \sigma^2/2)(T - t)}{\sigma \sqrt{T - t}} \]
\[ \mathcal{V} = \frac{S \sqrt{T - t}}{\sqrt{2\pi}} e^{-w^2/2} \]
Example (2 of 2)

\[ S = 200 \quad \sigma = 0.30 \quad T = 3/12 \]

\[ K = 195 \quad r = 0.0625 \quad t = 0 \]

Using these values

\[ w = \frac{\ln(S/K) + (r + \sigma^2/2)(T - t)}{\sigma \sqrt{T - t}} \approx 0.347952 \]

\[ \mathcal{V} = \frac{S \sqrt{T - t}}{\sqrt{2\pi}} e^{-w^2/2} \]
Example (2 of 2)

\[
S = 200 \quad \sigma = 0.30 \quad T = 3/12 \\
K = 195 \quad r = 0.0625 \quad t = 0
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Using these values

\[
w = \frac{\ln(S/K) + (r + \sigma^2/2)(T - t)}{\sigma \sqrt{T - t}} \approx 0.347952
\]

\[
V = \frac{S \sqrt{T - t}}{\sqrt{2\pi}} e^{-w^2/2} \approx 37.5509
\]
Example (2 of 2)

\[ S = 200 \quad \sigma = 0.30 \quad T = \frac{3}{12} \]
\[ K = 195 \quad r = 0.0625 \quad t = 0 \]

Using these values

\[
\begin{align*}
 w &= \frac{\ln(S/K) + (r + \sigma^2/2)(T - t)}{\sigma \sqrt{T - t}} \\
 &\approx 0.347952 \\

\mathcal{V} &= \frac{S \sqrt{T - t}}{\sqrt{2\pi}} e^{-w^2/2} \approx 37.5509
\end{align*}
\]

Using the linear approximation,

\[
dP = \mathcal{V} d\sigma = (37.5509)(0.01) = 0.375509.\]
Rho $\rho$ is the partial derivative with respect to the risk-free interest rate $r$. 

$$
C = S\phi(w) - Ke^{-r(T-t)}\phi(w - \sigma \sqrt{T-t})
$$

$$
\frac{\partial C}{\partial r} = S\phi'(w) \frac{\partial w}{\partial r} + Ke^{-r(T-t)}\phi(w - \sigma \sqrt{T-t})
$$

$$
- Ke^{-r(T-t)}\phi'(w - \sigma \sqrt{T-t}) \frac{\partial w}{\partial r}
$$
Rho is the partial derivative with respect to the risk-free interest rate $r$.

\[
C = S \phi(w) - Ke^{-r(T-t)} \phi(w - \sigma \sqrt{T-t})
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\frac{\partial C}{\partial r} = S \phi'(w) \frac{\partial w}{\partial r} + K(T-t) e^{-r(T-t)} \phi(w - \sigma \sqrt{T-t})
\]

\[
- Ke^{-r(T-t)} \phi'(w - \sigma \sqrt{T-t}) \frac{\partial w}{\partial r}
\]

\[
= K(T-t) e^{-r(T-t)} \phi(w - \sigma \sqrt{T-t})
\]
Starting with the Put-Call Parity formula:

\[ P + S = C + Ke^{-r(T-t)} \]
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\[ P + S = C + Ke^{-r(T-t)} \]

\[ \frac{\partial}{\partial r} (P + S) = \frac{\partial}{\partial r} (C + Ke^{-r(T-t)}) \]
Starting with the Put-Call Parity formula:

\[ P + S = C + Ke^{-r(T-t)} \]

\[ \frac{\partial}{\partial r} (P + S) = \frac{\partial}{\partial r} (C + Ke^{-r(T-t)}) \]

\[ \frac{\partial P}{\partial r} = \frac{\partial C}{\partial r} - K(T - t)e^{-r(T-t)} \]

\[ \frac{\partial P}{\partial r} = -K(T - t)e^{-r(T-t)}\phi(\sigma\sqrt{T - t - w}) \]
Example

Consider a three-month European put option on a stock whose current value is $200 and whose volatility is 30%. The option has a strike price of $195 and the risk-free interest rate is 6.25%.

1. Find the rho of the option.

2. If the interest rate increases to 7.00%, approximate the change in the value of the put.
\[ S = 200 \quad \sigma = 0.30 \quad T = 3/12 \]
\[ K = 195 \quad r = 0.0625 \quad t = 0 \]
Example (2 of 2)

\[
S = 200 \quad \sigma = 0.30 \quad T = 3/12 \\
K = 195 \quad r = 0.0625 \quad t = 0
\]

Using these values

\[
w = \ln(S/K) + (r + \sigma^2/2)(T - t) \\
\rho = -K(T - t)e^{-r(T-t)}\phi(\sigma\sqrt{T - t - w})
\]

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\rho = -K(T - t)e^{-r(T-t)}\phi(\sigma\sqrt{T - t - w})
\]
Example (2 of 2)

\[ S = 200 \quad \sigma = 0.30 \quad T = \frac{3}{12} \]
\[ K = 195 \quad r = 0.0625 \quad t = 0 \]

Using these values

\[ w = \frac{\ln(S/K) + (r + \sigma^2/2)(T - t)}{\sigma \sqrt{T - t}} \approx 0.347952 \]
\[ \rho = -K(T - t)e^{-r(T-t)}\phi(\sigma \sqrt{T-t} - w) \]
\[ S = 200 \quad \sigma = 0.30 \quad T = 3/12 \]
\[ K = 195 \quad r = 0.0625 \quad t = 0 \]

Using these values

\[ w = \ln(S/K) + \frac{(r + \sigma^2/2)(T - t)}{\sigma \sqrt{T - t}} \approx 0.347952 \]
\[ \rho = -K(T - t)e^{-r(T-t)}\phi(\sigma \sqrt{T - t} - w) \approx -20.2315 \]
Example (2 of 2)

\[ S = 200 \quad \sigma = 0.30 \quad T = 3/12 \]
\[ K = 195 \quad r = 0.0625 \quad t = 0 \]

Using these values

\[ w = \frac{\ln(S/K) + (r + \sigma^2/2)(T - t)}{\sigma \sqrt{T - t}} \approx 0.347952 \]
\[ \rho = -K(T - t)e^{-r(T-t)}\phi(\sigma \sqrt{T - t} - w) \approx -20.2315 \]

Using the linear approximation,

\[ dP = \rho dr = (-20.2315)(0.0075) = -0.151737. \]