The Theory of Interest
An Undergraduate Introduction to Financial Mathematics

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**Definition**

**Interest** is money paid by a bank or other financial institution to an investor or depositor in exchange for the use of the depositor’s money.

Amount of interest is (usually) a fraction (called the interest rate) of the initial amount deposited called the **principal amount**.
Notation:

- $r$: interest rate per unit time
- $P$: principal amount
- $A$: amount due (account balance)
- $t$: time

These quantities are related through the equation:

$$A = P(1 + rt).$$
Once credited to the investor, the interest may be kept by the investor, and may earn interest itself.

If interest is credited once per year, then after $t$ years the amount due is

$$A = P(1 + r)^t.$$
If a portion of the interest is credited after a fraction of a year, then the interest is said to be **compounded**.

If there are $n$ **compounding periods** per year, then in $t$ years the amount due is

$$A = P \left(1 + \frac{r}{n}\right)^{nt}.$$
Example

Suppose an account earns 5.75% annually compounded monthly. If the principal amount is $3104 what is the amount due after three and one-half years?
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Solution:

\[ A = P \left(1 + \frac{r}{n}\right)^{tn} \]

\[ = 3104 \left(1 + \frac{0.0575}{12}\right)^{(3.5)(12)} \]

\[ \approx 3794.15 \]
Example
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Solution:

\[ A = P(1 + rt) \]
\[ = 3104(1 + 0.0575(3.5)) \]
\[ \approx 3728.68 \]
Effective Interest Rate

Definition

The annual interest rate equivalent to a given compound interest rate is called the **effective interest rate**.

\[
re = \left(1 + \frac{r}{n}\right)^n - 1
\]
Example

Suppose an account earns 5.75% annually compounded monthly. What is the effective interest rate?
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\[ r_e = \left(1 + \frac{r}{n}\right)^n - 1 \]

\[ = \left(1 + \frac{0.0575}{12}\right)^{12} - 1 \]

\[ \approx 0.0590398 \]
Continuous Compounding

What happens as we increase the frequency of compounding?

\[ A = \lim_{n \to \infty} P \left( 1 + \frac{r}{n} \right)^{nt} \]
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**Definition**

The amount due for **continuously compounded interest** is

\[ A = Pe^{rt} \]
Example

Suppose $3585 is deposited in an account which pays interest at an annual rate of 6.15% compounded continuously.

1. Find the amount due after two and one half years.
2. Find the equivalent annual effective simple interest rate.
1 Amount due:

\[ A = Pe^{rt} = 3585e^{0.0615(2.5)} \approx 4180.82 \]

2 Effective rate:

\[ r_e = e^r - 1 = e^{0.0615} - 1 \approx 0.0634305 \]
How do we rationally compare amounts of money paid at different times in an interest-bearing environment?
Present Value

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Definition

The **present value** of $A$, an amount due $t$ years from now subject to an interest rate $r$ is the principal amount $P$ which must be invested now so that $t$ years from now the accumulated principal and interest total $A$. 
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\[
P = A \left(1 + \frac{r}{n}\right)^{-nt} \quad \text{(discrete compounding)}
\]
Present Value

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$$P = A \left(1 + \frac{r}{n}\right)^{-nt} \quad \text{(discrete compounding)}$$

$$P = Ae^{-rt} \quad \text{(continuous compounding)}$$
Example

Suppose an investor will receive payments at the end of the next six years in the amounts shown in the table.

<table>
<thead>
<tr>
<th>Year</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Payment</td>
<td>465</td>
<td>233</td>
<td>632</td>
<td>365</td>
<td>334</td>
<td>248</td>
</tr>
</tbody>
</table>

If the interest rate is 3.99% compounded monthly, what is the present value of the investments?
Solution:

\[ P = \sum_{i=1}^{6} \left( A_i \left( 1 + \frac{0.0399}{12} \right)^{-12i} \right) \]

\[ \approx 2003.01 \]
Geometric Series

Theorem

If $a \neq 1$ then

$$S = 1 + a + a^2 + \cdots + a^n = \frac{1 - a^{n+1}}{1 - a}.$$
**Theorem**

If \( a \neq 1 \) then

\[
S = 1 + a + a^2 + \cdots + a^n = \frac{1 - a^{n+1}}{1 - a}.
\]

**Proof.**

Let \( S = 1 + a + a^2 + \cdots + a^n \) then

\[
aS = a + a^2 + \cdots + a^n + a^{n+1}
\]

and

\[
S - aS = (1 + a + a^2 + \cdots + a^n) - (a + a^2 + \cdots + a^n + a^{n+1})
\]

\[
S(1 - a) = 1 - a^{n+1}
\]

\[
S = \frac{1 - a^{n+1}}{1 - a}
\]
Suppose a loan of amount $P$ will be paid back discretely ($n$ times per year) over $t$ years. The unpaid portion of the loan is charged interest at annual rate $r$ compounded $n$ times per year. What is the discrete payment $x$?
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**Hint:** the present value of all the payments should equal the amount borrowed.
If the first payment must be made at the end of the first compounding period, then the present value of all the payments is

\[ x(1 + \frac{r}{n})^{-1} + x(1 + \frac{r}{n})^{-2} + \cdots + x(1 + \frac{r}{n})^{-nt} \]

\[ = x(1 + \frac{r}{n})^{-1} \frac{1 - (1 + \frac{r}{n})^{-nt}}{1 - (1 + \frac{r}{n})^{-1}} \]

\[ = x \frac{1 - (1 + \frac{r}{n})^{-nt}}{\frac{r}{n}} \]

Thus

\[ P = x \frac{n}{r} \left( 1 - \left[ 1 + \frac{r}{n} \right]^{-nt} \right) \]
Example

If a person borrows $25,000 for five years at an interest rate of 4.99% compounded monthly and makes equal monthly payments, what is the monthly payment?
Solution:

\[ x = P \frac{r}{n} \left( 1 - \left[ 1 + \frac{r}{n} \right]^{-nt} \right)^{-1} \]

\[ = 25000 \left( \frac{0.0499}{12} \right) \left( 1 - \left[ 1 + \frac{0.0499}{12} \right]^{-\left(12\right)(5)} \right)^{-1} \]

\[ \approx 471.67 \]
Example

Suppose a person is 25 years of age now and plans to retire at age 65. For the next 40 years they plan to invest a portion of their monthly income in securities which earn interest at the rate of 10% compounded monthly. After retirement the person plans on receiving a monthly payment (an annuity) in the absolute amount of $1500 for 30 years. How much should be set aside monthly for retirement?
Solution: The present value of all funds invested for retirement should equal the present value of all funds taken out during retirement.

\[
x \sum_{i=1}^{480} \left(1 + \frac{0.10}{12}\right)^{-i} = 1500 \sum_{i=481}^{840} \left(1 + \frac{0.10}{12}\right)^{-i}
\]

\[
= 1500 \left(1 + \frac{0.10}{12}\right)^{-480} \sum_{i=1}^{360} \left(1 + \frac{0.10}{12}\right)^{-i}
\]

\[
x = \frac{1500 \left(1 + \frac{0.10}{12}\right)^{-480} \sum_{i=1}^{360} \left(1 + \frac{0.10}{12}\right)^{-i}}{\sum_{i=1}^{480} \left(1 + \frac{0.10}{12}\right)^{-i}}
\]

\[
\approx 27.03
\]
An increase in the amount of money in circulation without a commensurate increase in the amount of available goods is a condition known as inflation. Thus relative to the supply of goods, the value of the currency is decreased.
Adjusting for Inflation

Definition

An increase in the amount of money in circulation without a commensurate increase in the amount of available goods is a condition known as inflation. Thus relative to the supply of goods, the value of the currency is decreased.

How does inflation (measured at an annual rate $i$) affect the value of deposits earning interest?
Suppose at the current time one unit of currency will purchase one unit of goods.
Inflation-adjusted Interest Rate

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- Invested in savings, that one unit of currency has a future value (in one year) of $1 + r$. 
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In one year the unit of goods will require $1 + i$ units of currency for purchase.
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- In one year the unit of goods will require $1 + i$ units of currency for purchase.
- The difference $(1 + r) - (1 + i) = r - i$ will be the real rate of growth in the unit of currency invested now.
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Invested in savings, that one unit of currency has a future value (in one year) of $1 + r$.

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The difference $(1 + r) - (1 + i) = r - i$ will be the real rate of growth in the unit of currency invested now.

This return on saving will not be earned until one year from now. The present value of $r - i$ under inflation rate $i$ is

$$ r_i = \frac{r - i}{1 + i}. $$
Example

Suppose a person is 25 years of age now and plans to retire at age 65. For the next 40 years they plan to invest a portion of their monthly income in securities which earn interest at the rate of 10% compounded monthly. After retirement the person plans on receiving a monthly payment (an annuity) in the absolute amount of $1500 for 30 years. How much should be set aside monthly for retirement if the annual inflation rate is 3%?
Solution: The inflation adjusted return on saving is

\[ r_i = \frac{r - i}{1 + i} = \frac{0.10 - 0.03}{1 + 0.03} \approx 0.0679612. \]

Using this value in place of \( r \) in the previous example we have

\[ x = \frac{1500 \left( 1 + \frac{0.0679612}{12} \right)^{-480} \sum_{i=1}^{360} \left( 1 + \frac{0.0679612}{12} \right)^{-i}}{\sum_{i=1}^{480} \left( 1 + \frac{0.0679612}{12} \right)^{-i}} \approx 92.84. \]
Example

Suppose a person takes out an mortgage loan in the amount of \( L \) and will make \( n \) equal monthly payments of amount \( x \) where the annual interest rate is \( r \) compounded monthly.

1. Express \( x \) as a function of \( L \), \( r \), and \( n \).
2. After the \( j \)th month, how much of the original amount borrowed remains?
3. How much of the \( j \)th payment goes to interest and how much goes to pay down the amount borrowed?
Mortgage Example (2 of 4)

The sum of the present values of all the payments must equal the amount loaned.

\[
L = \sum_{i=1}^{n} \frac{x}{(1 + r/12)^i}
\]

\[
= x \sum_{i=1}^{n} (1 + r/12)^{-i}
\]

\[
= x(1 + r/12)^{-1} \sum_{i=0}^{n-1} (1 + r/12)^{-i}
\]

\[
= x(1 + r/12)^{-1} \frac{1 - (1 + r/12)^{-n}}{1 - (1 + r/12)^{-1}}
\]

\[
= \frac{x [1 - (1 + r/12)^{-n}]}{(1 + r/12)^{-n} - 1}
\]

\[
= \frac{12x}{r} \left[ 1 - (1 + r/12)^{-n} \right]
\]
The outstanding balance on the loan immediately after the $j$th monthly payment will be the sum of the present values of the remaining payments. Let $L_j$ denote the outstanding balance immediately after the $j$th payment, then

$$
L_j = \sum_{i=1}^{n-j} \frac{x}{(1 + \frac{r}{12})^i}
$$

$$
= x(1 + r/12)^{-1} \sum_{i=0}^{n-j-1} (1 + \frac{r}{12})^{-i}
$$

$$
= x(1 + r/12)^{-1} \frac{1 - (1 + r/12)^{-n+j}}{1 - (1 + r/12)^{-1}}
$$

$$
= x \left[ 1 - (1 + r/12)^{-n+j} \right] \frac{1}{(1 + r/12) - 1}
$$

$$
= \frac{12x}{r} \left[ 1 - \left(1 + \frac{r}{12}\right)^{-n+j} \right].
$$
If $l_j$ represents the amount of interest in the $j$th payment, then

$$l_j = L_{j-1}(r/12) = x \left[ 1 - \left( 1 + \frac{r}{12} \right)^{-n+j-1} \right].$$

The amount of principal repaid in the $j$th payment is

$$P_j = x - l_j = x \left( 1 + \frac{r}{12} \right)^{-n+j-1}.$$
Definition

If interest is compounded continuously at a time-dependent rate \( r(t) \), the function \( r(t) \) is referred to as the \textbf{spot rate}. 
Continuously Varying Interest Rates (1 of 2)

Definition

If interest is compounded continuously at a time-dependent rate \( r(t) \), the function \( r(t) \) is referred to as the **spot rate**.

- Suppose the amount due at \( t = 0 \) is \( A(0) = 1 \).
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- Suppose the amount due at \( t = 0 \) is \( A(0) = 1 \).
- The amount due at time \( t \) is \( A(t) \) and if \( \Delta t \) is small then

\[
\frac{A(t + \Delta t) - A(t)}{\Delta t} \approx r(t)A(t)
\]

\[
A'(t) = r(t)A(t).
\]
Amount due at time \( t > 0 \) on a unit deposit:

\[
A(t) = e^{\int_0^t r(s) \, ds}
\]
Amount due at time $t > 0$ on a unit deposit:

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Present value of a unit due at time $t > 0$:

$$P(t) = e^{-\int_0^t r(s) \, ds}$$
Amount due at time \( t > 0 \) on a unit deposit:

\[
A(t) = e^{\int_0^t r(s) \, ds}
\]

Present value of a unit due at time \( t > 0 \):

\[
P(t) = e^{-\int_0^t r(s) \, ds}
\]

**Definition**

The average of the spot rate over the interval \([0, t]\)

\[
\bar{r}(t) = \frac{1}{t} \int_0^t r(s) \, ds
\]

is called the **yield curve**.
Example

Suppose the spot rate is $r(t) = \frac{r_1}{1 + t} + \frac{r_2 t}{1 + t}$.

1. Find the yield curve $\bar{r}(t)$.
2. Find the present value of a unit due at time $t > 0$. 

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The Theory of Interest
Yield curve:

\[
\bar{r}(t) = \frac{1}{t} \int_{0}^{t} \left( \frac{r_1}{1 + s} + \frac{r_2 s}{1 + s} \right) \, ds
\]

\[
= \frac{r_1}{t} \ln(1 + t) + \frac{r_2}{t} (t - \ln(1 + t))
\]

\[
= r_2 + \frac{r_1 - r_2}{t} \ln(1 + t)
\]
Present value of a unit amount:

\[
P(t) = e^{-\int_0^t r(s) \, ds} = e^{-t \bar{r}(t)} = e^{-t \left( r_2 + \frac{r_1 - r_2}{t} \ln(1+t) \right)} = e^{-r_2 t - (r_1 - r_2) \ln(1+t)} = (1 + t)^{r_2 - r_1} e^{-r_2 t}
\]
If an investment of amount $P$ now receives an amount due of $A$ one time unit from now, the rate of return (denoted $r$) is the equivalent interest rate so that the present value of $A$ is $P$.

$$P = A(1 + r)^{-1}$$
Example

If you loan a friend $100 today with the understanding that they will pay you back $110 in one year’s time, what is the rate of return?
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Solution:

\[
P = A(1 + r)^{-1}
\]

100 = 110(1 + r)^{-1}

\[
1 + r = \frac{110}{100}
\]

\[
r = 0.10
\]
Suppose you invest an amount $P$ now and receive a sequence of positive payoffs $\{A_1, A_2, \ldots, A_n\}$ at regular intervals. The rate of return per period is the interest rate such that the present value of the sequence of payoffs is equal to the amount invested.

$$P = \sum_{i=1}^{n} A_i (1 + r)^{-i}.$$
Example

Suppose you loan a friend $100 with the agreement that they will pay you at the end of each year for the next five years amounts \{21, 22, 23, 24, 25\}. Find the annual rate of return.
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Solution:

\[
100 = \frac{21}{1 + r} + \frac{22}{(1 + r)^2} + \frac{23}{(1 + r)^3} + \frac{24}{(1 + r)^4} + \frac{25}{(1 + r)^5}
\]

\[
r \approx 0.0470299
\]

The solution to the equation is approximated using Newton’s Method with an initial approximation of 0.03.