Measures of Position
MATH 130, *Elements of Statistics I*

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A **measure of position** gives you some idea of

1. where particular data values would rank in an ordering of a data set, or

2. where a data value falls with respect to the mean of the sample or population.
z-Scores

Definition
The \textbf{z-score} represents the distance that a data value is from the mean in terms of standard deviations.

Population z-score: \[ z = \frac{x - \mu}{\sigma} \]

Sample z-score: \[ z = \frac{x - \bar{x}}{s} \]

The z-score has no units. It has a mean of 0 and a standard deviation of 1.
Example

Consider the data:

\[
\begin{array}{ccccccccccc}
3 & 4 & 5 & 5 & 5 & 5 & 5 & 5 & 6 & 7 & 7 \\
7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 8 & 8 & 8 \\
8 & 8 & 8 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 \\
10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 11 & 11 & 11 \\
11 & 11 & 11 & 11 & 11 & 11 & 12 & 12 & 12 & 12 & 12 \\
12 & 12 & 12 & 13 & 13 & 13 & 15 & 17 & 18 & 18 &
\end{array}
\]

We can calculate $\mu = 9.5$ and $\sigma = 3.1$. 

Example

Consider the data:

<table>
<thead>
<tr>
<th>3</th>
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</tbody>
</table>

We can calculate $\mu = 9.5$ and $\sigma = 3.1$. Thus the $z$-score corresponding to a data value of 11 is

$$z = \frac{11 - 9.5}{3.1} = 0.48.$$
Example

If $\mu = 9.5$ and $\sigma = 3.1$, find the $z$-scores corresponding to

1. $x = 7$

$z = \frac{x - \mu}{\sigma} = \frac{7 - 9.5}{3.1} = -\frac{2}{3.1}$

$z = -0.65$ (rounded to two decimal places)

2. $x = 13$

$z = \frac{x - \mu}{\sigma} = \frac{13 - 9.5}{3.1} = 1.13$ (rounded to two decimal places)

3. $x = 20$

$z = \frac{x - \mu}{\sigma} = \frac{20 - 9.5}{3.1} = 3.29$ (rounded to two decimal places)
Example

If $\mu = 9.5$ and $\sigma = 3.1$, find the $z$-scores corresponding to

1. $x = 7$

$$ z = \frac{7 - 9.5}{3.1} = \frac{-2.5}{3.1} = -0.81 $$

2. $x = 13$

3. $x = 20$
Example

If $\mu = 9.5$ and $\sigma = 3.1$, find the $z$-scores corresponding to

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$$z = \frac{13 - 9.5}{3.1} = \frac{3.5}{3.1} = 1.13$$

3. $x = 20$
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1. \( x = 7 \)

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2. \( x = 13 \)

\[
z = \frac{x - \mu}{\sigma} = \frac{13 - 9.5}{3.1} = \frac{3.5}{3.1} = 1.13
\]

3. \( x = 20 \)

\[
z = \frac{x - \mu}{\sigma} = \frac{20 - 9.5}{3.1} = \frac{10.5}{3.1} = 3.39
\]
Example

If $\mu = 9.5$ and $\sigma = 3.1$, find the data values corresponding to $z$-scores of

1. $z = -2.0$

2. $z = 1.51$

3. $z = 2.575$
Example

If $\mu = 9.5$ and $\sigma = 3.1$, find the data values corresponding to $z$-scores of

1. $z = -2.0$

\[-2.0 = \frac{x - 9.5}{3.1}\]
\[x = (-2.0)(3.1) + 9.5 = 3.3\]

2. $z = 1.51$

3. $z = 2.575$
Example

If $\mu = 9.5$ and $\sigma = 3.1$, find the data values corresponding to $z$-scores of

1. $z = -2.0$

\[-2.0 = \frac{x - 9.5}{3.1}\]
\[x = (-2.0)(3.1) + 9.5 = 3.3\]

2. $z = 1.51$

\[1.51 = \frac{x - 9.5}{3.1} \implies x = 14.2\]

3. $z = 2.575$
Example
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2. $z = 1.51$

$$1.51 = \frac{x - 9.5}{3.1} \implies x = 14.2$$

3. $z = 2.575$

$$2.575 = \frac{x - 9.5}{3.1} \implies x = 17.5$$
Percentiles

Definition
The \textit{kth percentile}, denoted $P_k$, of a set of data divides the lower $k\%$ of a data set from the upper $(100 - k)\%$. There are 99 percentiles.
Percentiles

**Definition**
The *kth percentile*, denoted $P_k$, of a set of data divides the lower $k\%$ of a data set from the upper $(100 - k)\%$. There are 99 percentiles.

**Interpretation:** The data value at the 40th percentile separates the lower 40% of the data from the upper 60% of the data.
Determining the $k$th Percentile

1. Arrange the data in ascending order.
2. Compute an index $i$ using the formula:

$$i = \left( \frac{k}{100} \right) (n + 1)$$

where $k$ is the percentile and $n$ is the number of observations in the data set.
3. If $i$ is a whole number, the $k$th percentile is the $i$th data value. If $i$ is not a whole number, the $k$th percentile is the mean of the observations on either side of $i$. 
Example

Consider the data:

\[
\begin{array}{cccccccccccc}
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\]

Find $P_{23}$, $P_{45}$, $P_{75}$, and $P_{90}$. 
To find \( P_{23} \):

- Index, \( i = \left( \frac{23}{100} \right) (60 + 1) = 14.03 \) (not a whole number)
- The 14th and 15th values of the variable are respectively, 7 and 7.
- The mean of 7 and 7 is 7, thus \( P_{23} = 7 \).

To find \( P_{75} \):

- Index, \( i = \left( \frac{75}{100} \right) (60 + 1) = 45.75 \)
- The 45th and 46th values of the variable are 11, thus \( P_{75} = 11 \).
Solution

To find \( P_{23} \):

- Index, \( i = \left( \frac{23}{100} \right) (60 + 1) = 14.03 \) (not a whole number)
- The 14th and 15th values of the variable are respectively, 7 and 7.
- The mean of 7 and 7 is 7, thus \( P_{23} = 7 \).

To find \( P_{75} \):

- Index, \( i = \left( \frac{75}{100} \right) (60 + 1) = 45.75 \)
- The 45th and 46th values of the variable are 11, thus \( P_{75} = 11 \).
Finding the Percentile of a Given Data Value

1. Arrange the data in ascending order.
2. Use the following formula:

   \[ \text{Percentile of } x = \frac{\text{number of data values less than } x}{n} \times 100 \]

   Round to the nearest whole number.
Example

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</tbody>
</table>

Find the percentile scores of 8, 10, 15, and 18.
Solution (1 of 2)

Percentile of 8:
\[
\frac{17}{60} \times 100 = 28.3 \approx 28
\]
In other words, \( P_{28} = 8 \).
Solution (1 of 2)

Percentile of 8:

\[
\frac{17}{60} \times 100 = 28.3 \approx 28
\]

In other words, \( P_{28} = 8 \).

Percentile of 10:

\[
\frac{30}{60} \times 100 = 50
\]

In other words, \( P_{50} = 10 \).
Solution (2 of 2)

Percentile of 15:

\[
\frac{56}{60} \times 100 = 93.3
\]

In other words, \( P_{93} = 15 \).
Solution (2 of 2)

Percentile of 15:

\[
\frac{56}{60} \times 100 = 93.3
\]

In other words, \( P_{93} = 15 \).

Percentile of 18:

\[
\frac{58}{60} \times 100 = 96.7 \approx 97
\]

In other words, \( P_{97} = 18 \).
Quartiles divide data sets into fourths.

\[ Q_1 = P_{25} \]
\[ Q_2 = P_{50} = M = \bar{x} \]
\[ Q_3 = P_{75} \]
Example

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12 & 12 & 12 & 13 & 13 & 13 & 15 & 17 & 18 & 18 \\
\end{array}
\]

Find $Q_1$, $Q_2$, and $Q_3$. 
Solution

\[ i = \left( \frac{25}{100} \right) (60 + 1) = 15.25 \]

\[ Q_1 = P_{25} = \frac{7 + 7}{2} = 7 \]

\[ i = \left( \frac{50}{100} \right) (60 + 1) = 30.50 \]

\[ Q_2 = P_{50} = \frac{9 + 10}{2} = 9.5 \]

\[ i = \left( \frac{75}{100} \right) (60 + 1) = 45.75 \]

\[ Q_3 = P_{75} = \frac{11 + 11}{2} = 11 \]
Outliers

Extreme data values are called **outliers**. We may check for them using the following procedure.

1. Determine $Q_1$ and $Q_3$.
2. Compute the **interquartile range**:
   \[ \text{IQR} = Q_3 - Q_1 \]
3. Determine the **fences**:
   \[ \text{lower fence} = Q_1 - 1.5(\text{IQR}) \]
   \[ \text{upper fence} = Q_3 + 1.5(\text{IQR}) \]
4. An outlier is a data value smaller than the lower fence or larger than the upper fence.
Determine if there are any outliers in the following data set.

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<td>3.02</td>
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*Hint:* $Q_1 = 6.25$ and $Q_3 = 10.73$. 
Solution

1. $Q_1 = 6.25$ and $Q_3 = 10.73$.
2. Interquartile range: $IQR = 10.73 - 6.25 = 4.48$.
3. Fences:
   
   \[
   \text{lower fence} = 6.25 - (1.5)(4.48) = -0.47 \\
   \text{upper fence} = 10.73 + (1.5)(4.48) = 17.45
   \]

4. There are no outliers in the data set.