Functions: Polynomial, Rational, Exponential

MATH 151 *Calculus for Management*

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Objectives

In this lesson we will learn to:

▶ identify polynomial expressions,
▶ classify certain polynomials as monomials, binomials, or trinomials,
▶ add, subtract, multiply, and factor polynomials,
▶ define rational functions and find their domains, and
▶ define exponential functions.
Polynomials

Definition
A **polynomial** is function that can be written in the form

\[ f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0. \]

The **degree of a polynomial** is the largest of the degrees of its terms after like terms have been combined.

The coefficient of the term with the largest degree is called the **leading coefficient**.

Polynomials with one, two, or three terms are called **monomials**, **binomials**, and **trinomials** respectively.

Polynomials of degree 0 or 1 are called **linear**, of degree 2 are called **quadratic**, and of degree 3 are called **cubic**.
Polynomial Equations

Equations involving polynomials can be solved by grouping all terms to one side of the equation and factoring.

Example

\[ 2x^3 - 4x^2 = 48x \]
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\[2x^3 - 4x^2 = 48x\]
\[2x^3 - 4x^2 - 48x = 0\]
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\[2x(x^2 - 2x - 24) = 0\]
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\[ 2x(x - 6)(x + 4) = 0 \]
\[ x = 0, \quad x = 6, \quad x = -4 \]
Rational Functions

Definition
A **rational function** is a quotient of polynomial functions:

\[ f(x) = \frac{N(x)}{D(x)} \]

where \( N(x) \) and \( D(x) \) are polynomials. The domain of a rational function is the set of real numbers for which the denominator is not zero.

Example
Find the domain of the following rational function.

\[ f(x) = \frac{x^2}{x^2 - 9} \]

**Domain:** \( \{ x \mid x \neq \pm 3 \} \)
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**Vertical and Horizontal Asymptotes**

**Definition**

1. The line \( x = a \) is a **vertical asymptote** of the graph of \( f \) if

\[
f(x) \to \infty \quad \text{or} \quad f(x) \to -\infty
\]

as \( x \to a \), either from the right or from the left.

2. The line \( y = b \) is a **horizontal asymptote** of the graph of \( f \) if

\[
f(x) \to b
\]

as \( x \to \infty \) or \( x \to -\infty \).
Illustration

\[ f(x) = \frac{x^2}{x^2 - 9} \]
Example

Find the horizontal and vertical asymptotes (if any) for the following function.

\[ f(x) = \frac{6x^2 - 11x + 3}{6x^2 - 7x - 3} \]
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\[ f(x) = \frac{6x^2 - 11x + 3}{6x^2 - 7x - 3} \]

\[ f(x) = \frac{(3x - 1)(2x - 3)}{(3x + 1)(2x - 3)} = \frac{3x - 1}{3x + 1} \]

if \( x \neq 3/2 \). The domain is the set of real numbers \( \{ x \mid x \neq 3/2, x \neq -1/3 \} \).

There is a vertical asymptote at \( x = -\frac{1}{3} \).

There is a horizontal asymptote at \( y = \frac{6}{6} = 1 \).
Exponential Functions

Definition
If $b > 0$ and $b \neq 1$, then the exponential function with base $b$ is given by

$$f(x) = c \cdot b^{rx}$$

where $c$ and $r$ are real number constants.
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where $c$ and $r$ are real number constants.

**Note:** the independent variable $x$ is in the exponent position and the base is a real number.
If $f(x) = b^x$ with $b > 1$, then

- **Domain**: $(-\infty, \infty)$
- **Range**: $(0, \infty)$
- **Intercept**: $(0, 1)$
- **Invertible**
- **Always increasing**
If \( f(x) = b^{-x} \) with \( b > 1 \), then

- **Domain**: \((-\infty, \infty)\)
- **Range**: \((0, \infty)\)
- **Intercept**: \((0, 1)\)
- **Invertible**
- **Always decreasing**
If the rule for calculating the values of a function is given in several parts depending on the portion of the domain the independent variable lies in, the function is said to be **piecewise-defined**.

**Example**

The following function is piecewise linear.

\[
f(x) = \begin{cases} 
5 - 2x & \text{if } x \geq 2, \\
 x + 3 & \text{if } x < 2.
\end{cases}
\]
Example: Income Tax Function

Federal income taxes are “progressive”, meaning that they take a higher percentage of higher incomes. For example the 2010 federal income tax for a single taxpayer whose taxable income was not more than $82,400 was determined by a three-part rule: 10% of income up to $8375, plus 15% of any amount over $8375 up to $34,000, plus 25% of any amount over $34,000 up to $82,400. For an income of $x$ dollars, the tax $T(x)$ may be expressed as:

$$ T(x) = \begin{cases} 
0.10x & \text{if } 0 \leq x \leq 8375, \\
837.50 + 0.15(x - 8375) & \text{if } 8375 < x \leq 34,000, \\
4681.25 + 0.25(x - 34000) & \text{if } 34,000 < x \leq 82,400. 
\end{cases} $$
Illustration

$q(\tilde{n})$

RMMM

NMMM

RM

NRMM

QMMM

QMMM

SMMM

UMMM

\tilde{n}
Example

Use your graphing calculator to graph the following function.

\[ f(x) = \begin{cases} 
2x + 3 & \text{if } x < 1, \\
6 - x & \text{if } 1 \leq x \leq 4, \\
(x - 4)^2 & \text{if } x > 4. 
\end{cases} \]