Antiderivatives
MATH 161 *Calculus I*

J. Robert Buchanan

Department of Mathematics

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Motivation

Many times a mathematical description of a situation specifies the rates of change of quantities.

Example
An object initially at altitude $y_0$ and traveling at velocity $v_0$ is allowed to move under the influence of gravitational acceleration $g$. Find the position of the object at any time $t$. 

\begin{align*}
\frac{d^2y}{dt^2} &= g \\
\frac{dy}{dt} &= gt + v_0 \\
y(t) &= \frac{1}{2}gt^2 + v_0t + y_0
\end{align*}
Many times a mathematical description of a situation specifies the rates of change of quantities.

**Example**

An object initially at altitude $y_0$ and traveling at velocity $v_0$ is allowed to move under the influence of gravitational acceleration $g$. Find the position of the object at any time $t$.

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\begin{align*}
y''(t) &= g \\
y'(t) &= g \, t + v_0 \\
y(t) &= \frac{1}{2} g \, t^2 + v_0 \, t + y_0
\end{align*}
\]
Remarks

- If $f(x)$ is the derivative of $F(x)$, in other words if $F'(x) = f(x)$ then $F(x)$ is called an antiderivative of $f(x)$. 

Antidifferentiation can be thought of as the reverse process of differentiation. Antidifferentiation is the second major theme of calculus. Antidifferentiation can be thought of as a process of accumulation or summing.
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- **Antidifferentiation** can be thought of as the reverse process of differentiation.
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- If \( f(x) \) is the derivative of \( F(x) \), in other words if \( F'(x) = f(x) \) then \( F(x) \) is called an **antiderivative** of \( f(x) \).
- **Antidifferentiation** can be thought of as the reverse process of differentiation.
- Antidifferentiation is the second major theme of calculus.
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- **Antidifferentiation** can be thought of as the reverse process of differentiation.
- Antidifferentiation is the second major theme of calculus.
- Antidifferentiation can be thought of as a process of accumulation or summing.
Example

Find three different antiderivatives of $f(x) = x^4$. 

$F_1(x) = \frac{1}{5}x^5$

$F_2(x) = \frac{1}{5}x^5 + 7$

$F_3(x) = \frac{1}{5}x^5 - \frac{1}{10}$

Infinitely many more answers are possible.
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Infinitely many more answers are possible.
Theorem

Suppose that $F$ and $G$ are both antiderivatives of $f$ on interval $I$. Then $F(x) = G(x) + C$ for some constant $C$.

Remark: we proved this earlier in the section on the Mean Value Theorem.
Definition
Let $F$ be any antiderivative of $f$. The **indefinite integral** of $f(x)$ with respect to $x$ is defined by

$$\int f(x) \, dx = F(x) + C$$

where $C$ is called the **constant of integration**.
Examples

Find the indefinite integrals of the following functions.

✈️ \( f(x) = e^x \)

✈️ \( g(x) = \frac{1}{x} \text{ for } x > 0 \)

✈️ \( h(x) = \cos x \)

✈️ \( k(x) = \frac{1}{1 + x^2} \)
Examples

Find the indefinite integrals of the following functions.

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- $k(x) = \frac{1}{1 + x^2}$
  \[
  \int \frac{1}{1 + x^2} \, dx = \tan^{-1} x + C
  \]
Theorem (Power Rule)

For any rational number $r \neq -1$, 

$$\int x^r \, dx = \frac{1}{r+1} x^{r+1} + C.$$
Examples

Find the indefinite integrals of the following functions.

- \( f(x) = x^6 \)
  \[
  \int x^6 \, dx = \frac{1}{7}x^7 + C
  \]

- \( g(x) = \frac{1}{x^2} \)
  \[
  \int \frac{1}{x^2} \, dx = \int x^{-2} \, dx = -\frac{1}{x} + C
  \]

- \( h(x) = x^{1/2} \)
  \[
  \int x^{1/2} \, dx = \frac{2}{3}x^{3/2} + C
  \]

- \( k(x) = x^{-1/3} \)
  \[
  \int x^{-1/3} \, dx = \frac{3}{2}x^{2/3} + C
  \]
Examples

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  \int \frac{1}{x^2} \, dx = \int x^{-2} \, dx = \frac{1}{-1}x^{-1} + C = -\frac{1}{x} + C
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- $h(x) = x^{1/2}$
  \[
  \int x^{1/2} \, dx = \frac{1}{\frac{3}{2}}x^{3/2} + C = \frac{2}{3}x^{3/2} + C
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- \( k(x) = x^{-1/3} \)
  \[
  \int x^{-1/3} \, dx = \frac{1}{2/3}x^{2/3} + C = \frac{3}{2}x^{2/3} + C
  \]
Common Indefinite Integrals

\[ \int x^n \, dx = \frac{1}{n+1} x^{n+1} + C \]
if \( n \neq -1 \)

\[ \int \sin x \, dx = -\cos x + C \]

\[ \int \cos x \, dx = \sin x + C \]

\[ \int \sec^2 x \, dx = \tan x + C \]

\[ \int \csc^2 x \, dx = -\cot x + C \]

\[ \int \frac{1}{1 + x^2} \, dx = \tan^{-1} x + C \]

\[ \int \frac{1}{\sqrt{1 - x^2}} \, dx = \sin^{-1} x + C \]

\[ \int \frac{1}{|x| \sqrt{x^2 - 1}} \, dx = \sec^{-1} x + C \]

\[ \int \sec x \tan x \, dx = \sec x + C \]

\[ \int \csc x \cot x \, dx = -\csc x + C \]

\[ \int e^x \, dx = e^x + C \]

\[ \int e^{-x} \, dx = -e^{-x} + C \]
Theorem
Suppose that $f(x)$ and $g(x)$ have antiderivatives. Then, for any constants $a$ and $b$,

$$
\int (af(x) + bg(x)) \, dx = a \int f(x) \, dx + b \int g(x) \, dx.
$$
Examples

Evaluate the following indefinite integrals.

\[ \int (3 \cos x - \sin x) \, dx = 3 \sin x + \cos x + C \]

\[ \int (4x - 2x^{-3}) \, dx = 2x^2 + x - 2 + C \]

\[ \int \left( 2 \sec x \tan x + \frac{1}{\sqrt{x}} \right) \, dx = 2 \sec x + 2 \sqrt{x} + C \]
Examples

Evaluate the following indefinite integrals.

1. \[
\int (3 \cos x - \sin x) \, dx = 3 \sin x + \cos x + C
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2. \[
\int (4x - 2x^{-3}) \, dx
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Indefinite Integrals Involving the Natural Logarithm (1 of 2)

Theorem

If $x \neq 0$, \[ \frac{d}{dx} \ln |x| = \frac{1}{x}. \]
Theorem
If \( x \neq 0 \), \( \frac{d}{dx} \ln |x| = \frac{1}{x} \).

Proof.

- Suppose \( x > 0 \), then
  \[
  \frac{d}{dx} \ln |x| = \frac{d}{dx} \ln x = \frac{1}{x}.
  \]
Theorem

If \( x \neq 0 \), \( \frac{d}{dx} \ln |x| = \frac{1}{x} \).

Proof.

\( \boxed{\text{Suppose } x > 0, \text{ then}} \)

\[
\frac{d}{dx} \ln |x| = \frac{d}{dx} \ln x = \frac{1}{x}.
\]

\( \boxed{\text{Suppose } x < 0, \text{ then}} \)

\[
\frac{d}{dx} \ln |x| = \frac{d}{dx} \ln(-x) = \frac{1}{-x}(-x)' = \frac{1}{x}.
\]
Examples

Find the derivatives of the following functions.

- $f(x) = \ln |\cos x|$

- $g(x) = \ln |f(x)|$ if $f(x) \neq 0$. 
Examples

Find the derivatives of the following functions.

1. \( f(x) = \ln |\cos x| \)

   \[
   f'(x) = \frac{1}{\cos x} (\cos x)' = \frac{-\sin x}{\cos x} = -\tan x
   \]

2. \( g(x) = \ln |f(x)| \) if \( f(x) \neq 0 \).
Examples

Find the derivatives of the following functions.

- \( f(x) = \ln |\cos x| \)

\[
f'(x) = \frac{1}{\cos x} (\cos x)' = \frac{-\sin x}{\cos x} = -\tan x
\]

- \( g(x) = \ln |f(x)| \) if \( f(x) \neq 0 \).

\[
g'(x) = \frac{1}{f(x)} f'(x) = \frac{f'(x)}{f(x)}
\]
Corollary

If \( x \neq 0 \) then \( \int \frac{1}{x} \, dx = \ln |x| + C. \)
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If $x \neq 0$ then $\int \frac{1}{x} \, dx = \ln |x| + C.$

Corollary

If $f(x) \neq 0$ then $\int \frac{f'(x)}{f(x)} \, dx = \ln |f(x)| + C.$
Examples

Evaluate the following indefinite integrals.

1. \( \int \frac{\cos x}{\sin x} \, dx \)

2. \( \int \frac{2x}{x^2 - 1} \, dx \)

3. \( \int \frac{2x}{1 + x^2} \, dx \)
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Evaluate the following indefinite integrals.

1. \( \int \frac{\cos x}{\sin x} \, dx \)
   \[ \int \frac{\cos x}{\sin x} \, dx = \ln |\sin x| + C \]

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3. \( \int \frac{2x}{1 + x^2} \, dx \)
   \[ \int \frac{2x}{1 + x^2} \, dx = \ln (1 + x^2) + C \]
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Evaluate the following indefinite integrals.

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\[ \int \frac{\cos x}{\sin x} \, dx = \ln |\sin x| + C \]

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\[ \int \frac{2x}{1 + x^2} \, dx \]
Examples

Evaluate the following indefinite integrals.

1. \[ \int \frac{\cos x}{\sin x} \, dx = \ln |\sin x| + C \]

2. \[ \int \frac{2x}{x^2 - 1} \, dx = \ln |x^2 - 1| + C \]

3. \[ \int \frac{2x}{1 + x^2} \, dx = \ln |1 + x^2| + C = \ln(1 + x^2) + C \]
Homework

- Read Section 4.1
- Exercises: 1–47 odd