Area
MATH 161 *Calculus I*

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Motivation

Suppose \( f(x) \geq 0 \) is continuous on interval \([a, b]\). We would like to find the area under the graph of \( f(x) \), above the \( x \)-axis, and between \( x = a \) and \( x = b \).
Approximating Area

We can approximate the area under the curve using a regular partition and rectangles.

\[ A \approx \sum_{i=1}^{n} f(w_i) \Delta x \]
Regular Partition

Definition
A regular partition of \([a, b]\) is a subdivision of \([a, b]\) into \(n\) equal-sized pieces. The width of each subinterval is \(\Delta x = \frac{b-a}{n}\). The points in the partition are

\[
\{x_i = a + i\Delta x\} \quad \text{for } i = 0, 1, \ldots, n.
\]

- The right endpoints are \(x_i = a + i\Delta x\) for \(i = 1, 2, \ldots, n\).
- The midpoints are \(a + \left(i - \frac{1}{2}\right)\Delta x\) for \(i = 1, 2, \ldots, n\).
- The left endpoints are \(x_i = a + (i - 1)\Delta x\) for \(i = 1, 2, \ldots, n\).
Example

Approximate the area under the curve \( f(x) = 9 - x^2 \) on the interval \([-3, 3]\) using a partition of \( n = 6 \) subintervals and evaluating the function at the right endpoint of each subinterval.
Example

Approximate the area under the curve \( f(x) = 9 - x^2 \) on the interval \([-3, 3]\) using a partition of \( n = 6 \) subintervals and evaluating the function at the right endpoint of each subinterval.

\[
A \approx \sum_{i=1}^{6} f(-3 + i\Delta x)\Delta x
\]

\[
= \sum_{i=1}^{6} (9 - (-3 + i\Delta x)^2)\Delta x
\]

\[
= \sum_{i=1}^{6} (9 - (-3 + i(1))^2)(1)
\]

\[
= \sum_{i=1}^{6} (9 - (-3 + i)^2)
\]

\[
= 35
\]
Example

Approximate the area under the curve \( f(x) = 9 - x^2 \) on the interval \([-3, 3]\) using a partition of \( n = 6 \) subintervals and evaluating the function at the left endpoint of each subinterval.

\[
\sum_{i=1}^{6} f(-3 + (i-1)\Delta x)\Delta x = 35
\]
Example

Approximate the area under the curve $f(x) = 9 - x^2$ on the interval $[-3, 3]$ using a partition of $n = 6$ subintervals and evaluating the function at the left endpoint of each subinterval.

$$A \approx \sum_{i=1}^{6} f(-3 + (i - 1)\Delta x)\Delta x$$

$$= \sum_{i=1}^{6} (9 - (-3 + (i - 1)\Delta x)^2)\Delta x$$

$$= \sum_{i=1}^{6} (9 - (-3 + (i - 1)(1))^2)(1)$$

$$= \sum_{i=1}^{6} (9 - (-4 + i)^2)$$

$$= 35$$
Example

Approximate the area under the curve $f(x) = 9 - x^2$ on the interval $[-3, 3]$ using a partition of $n = 6$ subintervals and evaluating the function at the midpoint of each subinterval.

$$A \approx 6 \sum_{i=1}^{n} f(\left(-3 + (i - 1/2) \Delta x\right)) \Delta x = 6 \sum_{i=1}^{n} \left(9 - \left(-3 + (i - 1/2) \Delta x\right)^2\right) \Delta x = 6 \sum_{i=1}^{n} \left(9 - \left(-7/2 + i \Delta x\right)^2\right) \Delta x = 36.5.$$
Example

Approximate the area under the curve \( f(x) = 9 - x^2 \) on the interval \([-3, 3]\) using a partition of \( n = 6 \) subintervals and evaluating the function at the midpoint of each subinterval.

\[
A \approx \sum_{i=1}^{6} f \left( -3 + \left[ i - \frac{1}{2} \right] \Delta x \right) \Delta x
\]

\[
= \sum_{i=1}^{6} \left( 9 - (-3 + \left[ i - \frac{1}{2} \right] \Delta x)^2 \right) \Delta x
\]

\[
= \sum_{i=1}^{6} \left( 9 - (-3 + \left[ i - \frac{1}{2} \right] (1))^2 \right) (1)
\]

\[
= \sum_{i=1}^{6} \left( 9 - (-\frac{7}{2} + i)^2 \right)
\]

\[
= 36.5
\]
Example
Approximate the area under the curve $f(x) = e^{-3x}$ on the interval $[0, 2]$ using a partition of $n = 20$ subintervals and evaluating the function at the left endpoint of each subinterval.
Example

Approximate the area under the curve \( f(x) = e^{-3x} \) on the interval \([0, 2]\) using a partition of \( n = 20 \) subintervals and evaluating the function at the left endpoint of each subinterval.

\[
A \approx \sum_{i=1}^{20} f(0 + (i - 1)\Delta x) \Delta x \\
= \sum_{i=1}^{20} e^{-3(0+(i-1)\Delta x)} \Delta x \\
= \sum_{i=1}^{20} e^{-3((i-1)(0.1))} (0.1) \\
= \sum_{i=1}^{20} 0.1 \cdot e^{-0.3(i-1)} \\
= 0.384873
\]
Example

Approximate the area under the curve \( f(x) = x \sin x \) on the interval \([0, \pi]\) using a partition of \( n = 10 \) subintervals and evaluating the function at the midpoint of each subinterval.

\[
A \approx 10 \sum_{i=1}^{10} f(0 + \left[ i - 1 \right] \frac{\pi}{10}) \frac{\pi}{10} = 3.15455
\]
Example

Approximate the area under the curve $f(x) = x \sin x$ on the interval $[0, \pi]$ using a partition of $n = 10$ subintervals and evaluating the function at the midpoint of each subinterval.

$$A \approx \sum_{i=1}^{10} f \left( 0 + \left[ i - \frac{1}{2} \right] \Delta x \right) \Delta x$$

$$= \sum_{i=1}^{10} \left( 0 + \left[ i - \frac{1}{2} \right] \Delta x \right) \sin \left( 0 + \left[ i - \frac{1}{2} \right] \Delta x \right) \Delta x$$

$$= \sum_{i=1}^{10} \left( \left[ i - \frac{1}{2} \right] \frac{\pi}{10} \right) \sin \left( \left[ i - \frac{1}{2} \right] \frac{\pi}{10} \right) \frac{\pi}{10}$$

$$= 3.15455$$
Example

Approximate the area under the curve $f(x) = x \sin x$ on the interval $[0, \pi]$ using a partition of $n = 100$ subintervals and evaluating the function at the midpoint of each subinterval.

$$A \approx 100 \sum_{i=1}^{n} f(0 + \left(\frac{i-1}{2}\right) \Delta x) \Delta x = 100 \sum_{i=1}^{n} \left(\frac{i-1}{2}\right) \frac{\pi}{100} \sin \left(\frac{i-1}{2}\right) \frac{\pi}{100} \pi 100 = 3.14172$$
Example

Approximate the area under the curve \( f(x) = x \sin x \) on the interval \([0, \pi]\) using a partition of \( n = 100\) subintervals and evaluating the function at the midpoint of each subinterval.

\[
A \approx 100 \sum_{i=1}^{100} f \left( 0 + \left[ i - \frac{1}{2} \right] \Delta x \right) \Delta x \\
= 100 \sum_{i=1}^{100} \left( 0 + \left[ i - \frac{1}{2} \right] \Delta x \right) \sin \left( 0 + \left[ i - \frac{1}{2} \right] \Delta x \right) \Delta x \\
= 100 \sum_{i=1}^{100} \left( \left[ i - \frac{1}{2} \right] \frac{\pi}{100} \right) \sin \left( \left[ i - \frac{1}{2} \right] \frac{\pi}{100} \right) \frac{\pi}{100} \\
= 3.14172
\]
Exact Area Under a Curve

Definition
For a function $f$ defined on the interval $[a, b]$ if $f$ is continuous on $[a, b]$ and $f(x) \geq 0$ on $[a, b]$, the area $A$ under the curve where $y = f(x)$ is

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x.$$
Example (1 of 2)

Find the exact area under the graph of \( f(x) = 9 - x^2 \) on the interval \([-3, 3]\).
Example (1 of 2)

Find the exact area under the graph of $f(x) = 9 - x^2$ on the interval $[-3, 3]$.

Let $\Delta x = \frac{3 - (-3)}{n} = \frac{6}{n}$ and use right endpoint evaluation points.

$$c_i = -3 + i\Delta x = -3 + \frac{6i}{n}$$
Example (2 of 2)

\[
A = \lim_{n \to \infty} \sum_{i=1}^{n} f(c_i)\Delta x = \lim_{n \to \infty} \sum_{i=1}^{n} \left[ 9 - \left( -3 + \frac{6i}{n} \right)^2 \right] \frac{6}{n}
\]

\[
= \lim_{n \to \infty} \sum_{i=1}^{n} \left[ 9 - \left( 9 - \frac{36i}{n} + \frac{36i^2}{n^2} \right) \right] \frac{6}{n}
\]

\[
= \lim_{n \to \infty} \sum_{i=1}^{n} \left[ \frac{216i}{n^2} - \frac{216i^2}{n^3} \right]
\]

\[
= \lim_{n \to \infty} \left[ \frac{216}{n^2} \frac{n(n+1)}{2} - \frac{216}{n^3} \frac{n(n+1)(2n+1)}{6} \right]
\]

\[
= 108 - 72 = 36
\]
Riemann Sums

Definition
Let \( \{x_0, x_1, \ldots, x_n\} \) be a regular partition of the interval \([a, b]\) with \( x_i - x_{i-1} = \Delta x = \frac{b-a}{n} \) for \( i = 1, 2, \ldots n \). Pick points \( c_1, c_2, \ldots, c_n \), where \( x_{i-1} \leq c_i \leq x_i \) for \( i = 1, 2, \ldots n \) (these are called evaluation points). The Riemann sum for this partition and set of evaluation points is

\[
\sum_{i=1}^{n} f(c_i) \Delta x.
\]
Example

Let $f(x) = x^3 - 1$ on the interval $[-1, 1]$ and construct a table of Riemann sums for $n = 50, 100, \text{ and } 150$ and for left endpoint, right endpoint, and midpoint evaluation points.

<table>
<thead>
<tr>
<th>n</th>
<th>Left Endpoint</th>
<th>Midpoint</th>
<th>Right Endpoint</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>100</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>150</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Example

Let $f(x) = x^3 - 1$ on the interval $[-1, 1]$ and construct a table of Riemann sums for $n = 50, 100, \text{ and } 150$ and for left endpoint, right endpoint, and midpoint evaluation points.

<table>
<thead>
<tr>
<th></th>
<th>Left</th>
<th>Midpoint</th>
<th>Right</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = 50$</td>
<td>$-2.04$</td>
<td>$-2.0$</td>
<td>$-1.96$</td>
</tr>
<tr>
<td>$n = 100$</td>
<td>$-2.02$</td>
<td>$-2.0$</td>
<td>$-1.98$</td>
</tr>
<tr>
<td>$n = 150$</td>
<td>$-2.01333$</td>
<td>$-2.0$</td>
<td>$-1.98667$</td>
</tr>
</tbody>
</table>
Homework

- Read Section 4.3
- Exercises: 1–25 odd