A process or an item can be described as **continuous** if it exists without interruption.

The mathematical definition of continuity is more technical than intuitive, but is based on the concept of the limit.
Graphical Examples of Discontinuities (1 of 4)

\[ f(a) \] is undefined. The graph has a “hole” at \( x = a \).
Graphical Examples of Discontinuities (2 of 4)

$f(a)$ is defined but the \( \lim_{x \to a} f(x) \) does not exist. This is sometimes called a “jump discontinuity.”
Graphical Examples of Discontinuities (3 of 4)

\[ f(a) \text{ is defined and the } \lim_{x \to a} f(x) \text{ exists, but} \]
\[ \lim_{x \to a} f(x) \neq f(a). \]
The $\lim_{x \to a} f(x)$ does not exist (is not finite).
Definition of Continuity

Definition
A function $f$ is **continuous at** $x = a$ when
1. $f(a)$ is defined,
2. $\lim_{x \to a} f(x)$ exists, and
3. $\lim_{x \to a} f(x) = f(a)$.

Otherwise $f$ is said to be **discontinuous at** $x = a$. 
Examples (1 of 2)

Determine the values of $x$ for which $f(x) = \frac{x^2 - 1}{x - 1}$ is continuous.
Consider the piecewise-defined function

\[ f(x) = \begin{cases} 
  c e^x + 1 & \text{if } x < 0, \\
  \sin^{-1} \left( \frac{x}{2} \right) & \text{if } 0 \leq x \leq 2.
\end{cases} \]

Determine a value for the constant \( c \) which makes \( f(x) \) continuous at \( x = 0 \).
Removable Discontinuities

If we can eliminate a discontinuity at $x = a$ by redefining the function at $x = a$, then we say the discontinuity is **removable**.
Removable Discontinuities

If we can eliminate a discontinuity at $x = a$ by redefining the function at $x = a$, then we say the discontinuity is **removable**.

**Example**

If $f(x) = \frac{\sin x}{x}$ for $x \neq 0$, is the discontinuity at $x = 0$ removable? If so, define $f(0)$ so that $f(x)$ is continuous at $x = 0$. 
Solution

\[ f(x) = \frac{\sin x}{x} \]
Solution

\[ f(x) = \frac{\sin x}{x} \]

\[ g(x) = \begin{cases} 
\frac{\sin x}{x} & \text{if } x \neq 0, \\
1 & \text{if } x = 0. 
\end{cases} \]
Theorem

All polynomials are continuous for all $x$. Likewise $\sin x$, $\cos x$, $\tan^{-1} x$, and $e^x$ are continuous for all $x$. The radical function $\sqrt[n]{x}$ is continuous for all $x$ when $n$ is odd and continuous for all $x > 0$ when $n$ is even. Also $\ln x$ is continuous for all $x > 0$ and $\sin^{-1} x$ and $\cos^{-1} x$ are continuous for $-1 < x < 1$. 
Theorem
Suppose $f$ and $g$ are continuous at $x = a$, then

1. $(f \pm g)$ is continuous at $x = a$,
2. $(f \cdot g)$ is continuous at $x = a$,
3. $\left(\frac{f}{g}\right)$ is continuous at $x = a$, provided $g(a) \neq 0$. 
Determine the values of $x$ where $f(x) = \frac{4x}{x^2 + x - 2}$ is discontinuous.
Example

Determine the values of $x$ where $f(x) = \frac{4x}{x^2 + x - 2}$ is discontinuous.

**Strategy:** to find discontinuities, look for values of $x$ for which $f(x)$ is undefined, approaches an infinite limit, or has differing one-sided limits.
Example

Determine the values of $x$ where $f(x) = \frac{4x}{x^2 + x - 2}$ is discontinuous.

**Strategy**: to find discontinuities, look for values of $x$ for which $f(x)$ is undefined, approaches an infinite limit, or has differing one-sided limits.

Function $f(x)$ is continuous for $x \in (-\infty, -2) \cup (-2, 1) \cup (1, \infty)$. 
Theorem

Suppose that \( \lim_{{x \to a}} g(x) = L \) and that \( f \) is continuous at \( x = L \), then

\[
\lim_{{x \to a}} f(g(x)) = f \left( \lim_{{x \to a}} g(x) \right) = f(L).
\]
Composition of Functions and Continuity

**Theorem**
Suppose that \( \lim_{x \to a} g(x) = L \) and that \( f \) is continuous at \( x = L \), then
\[
\lim_{x \to a} f(g(x)) = f \left( \lim_{x \to a} g(x) \right) = f(L).
\]

**Corollary**
Suppose that \( g \) is continuous at \( x = a \) and that \( f \) is continuous at \( x = g(a) \), then \( f \circ g \) is continuous at \( x = a \).
Example

Find the values of $x$ for which $h(x) = \ln(4 - x^2)$ is continuous.
Example

Find the values of $x$ for which $h(x) = \ln(4 - x^2)$ is continuous.

- Let $h(x) = f(g(x))$ where $g(x) = 4 - x^2$ and $f(x) = \ln x$.
- Function $g(x)$ is continuous for all $x$ and $f(x)$ is continuous for all $x > 0$, thus $h(x)$ is continuous at all $x$ for which $g(x) > 0$. 
Example

Find the values of $x$ for which $h(x) = \ln(4 - x^2)$ is continuous.

- Let $h(x) = f(g(x))$ where $g(x) = 4 - x^2$ and $f(x) = \ln x$.
- Function $g(x)$ is continuous for all $x$ and $f(x)$ is continuous for all $x > 0$, thus $h(x)$ is continuous at all $x$ for which $g(x) > 0$.
- Function $h(x)$ is continuous for $x \in (-2, 2)$. 
Continuity on an Interval

**Definition**

If \( f \) is continuous at every point on an open interval of the form \((a, b)\) we say \( f \) is **continuous on** \((a, b)\). We say \( f \) is **continuous on** \([a, b]\) if \( f \) is continuous on \((a, b)\) and

\[
\lim_{x \to a^+} f(x) = f(a) \quad \text{and} \quad \lim_{x \to b^-} f(x) = f(b).
\]

If \( f \) is continuous on \((-\infty, \infty)\) then we say that \( f \) is **continuous everywhere**.
Example

Suppose the formula for the amount owed for state income tax is

\[ T(x) = \begin{cases} 
  0 & \text{if } x \leq 0, \\
  a + 0.12x & \text{if } 0 < x < 20,000, \\
  b + 0.16(x - 20,000) & \text{if } x \geq 20,000.
\end{cases} \]

where \( x \) is the income of the taxpayer. Determine values for the constants \( a \) and \( b \) that make the function \( T \) continuous everywhere.
Solution

- Since each piece of the piecewise-defined function is a polynomial, each piece is continuous. The "transition" points \((x = 0 \text{ and } x = 20,000)\) are the only possible locations of discontinuities.
Solution

► Since each piece of the piecewise-defined function is a polynomial, each piece is continuous. The “transition” points ($x = 0$ and $x = 20,000$) are the only possible locations of discontinuities.

► For $x = 0$,

\[
T(0) = 0
\]

\[
\lim_{{x \to 0^-}} T(x) = \lim_{{x \to 0^-}} 0 = 0
\]

\[
\lim_{{x \to 0^+}} T(x) = \lim_{{x \to 0^+}} (a + 0.12x) = a
\]

thus $T(x)$ is continuous at $x = 0$ if $a = 0$. 
Solution

Since each piece of the piecewise-defined function is a polynomial, each piece is continuous. The “transition” points \( x = 0 \) and \( x = 20,000 \) are the only possible locations of discontinuities.

For \( x = 0 \),

\[
\begin{align*}
T(0) &= 0 \\
\lim_{x \to 0^-} T(x) &= \lim_{x \to 0^-} 0 = 0 \\
\lim_{x \to 0^+} T(x) &= \lim_{x \to 0^+} (a + 0.12x) = a
\end{align*}
\]

thus \( T(x) \) is continuous at \( x = 0 \) if \( a = 0 \).

For \( x = 20,000 \),

\[
\begin{align*}
T(20,000) &= b \\
\lim_{x \to 20,000^-} T(x) &= \lim_{x \to 20,000^-} (a + 0.12x) = 2,400 \\
\lim_{x \to 20,000^+} T(x) &= \lim_{x \to 0^+} (b + 0.16(x - 20,000)) = b
\end{align*}
\]

thus \( T(x) \) is continuous at \( x = 20,000 \) if \( b = 2,400 \).
Theorem (Intermediate Value Theorem)

Suppose $f$ is continuous on the closed interval $[a, b]$ and $W$ is any number between $f(a)$ and $f(b)$. Then there is a number $c \in [a, b]$ for which $f(c) = W$. 
Corollary

Suppose $f$ is continuous on the closed interval $[a, b]$ and $f(a)$ and $f(b)$ have opposite signs (in other words $f(a)f(b) < 0$). Then there is at least one number $c \in (a, b)$ for which $f(c) = 0$. 
Corollary

Suppose $f$ is continuous on the closed interval $[a, b]$ and $f(a)$ and $f(b)$ have opposite signs (in other words $f(a)f(b) < 0$). Then there is at least one number $c \in (a, b)$ for which $f(c) = 0$.

This corollary forms the basis of the **Method of Bisections** for approximating the roots of equations.
Example (1 of 2)

Use the Method of Bisections to approximate the solution to the equation

$$2 \ln x - \sqrt{x} = 0.$$
**Example (2 of 2)**

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Homework

- Read Section 1.4
- Exercises: 1–43 odd