The Definite Integral
MATH 161 *Calculus I*

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We have seen that if $f(x) \geq 0$ and continuous on the interval $[a, b]$ the exact area under the graph of $f(x)$, above the $x$-axis, and between $x = a$ and $x = b$ is

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(c_i)\Delta x.$$
Background

We have seen that if $f(x) \geq 0$ and continuous on the interval $[a, b]$ the **exact area** under the graph of $f(x)$, above the $x$-axis, and between $x = a$ and $x = b$ is

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(c_i) \Delta x.$$ 

**Remark:** the limit makes sense even when $f(x) < 0$, but the value of the limit cannot be strictly interpreted as area.
Definite Integral

Definition
For any function $f$ defined on $[a, b]$, the **definite integral** of $f$ from $a$ to $b$ is

$$
\int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(c_i) \Delta x,
$$

whenever the limit exists and is the same for any choice of evaluation points $c_1, c_2, \ldots, c_n$. When the limit exists we say that $f$ is **integrable** on $[a, b]$. 

Notation

\[ \int_{a}^{b} f(x) \, dx \]

- \( a \): is called the **lower limit of integration**.
- \( b \): is called the **upper limit of integration**.
- \( f(x) \): is called the **integrand**.
- \( dx \): differential of \( x \), denotes the **variable of integration**.
Use the limit of a Riemann sum to compute the following definite integral exactly.

\[ \int_{0}^{3} (x^2 + 1) \, dx \]
Solution

\[
\int_0^3 (x^2 + 1) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} \left[ \left( \left( \frac{3i}{n} \right)^2 + 1 \right) \cdot \frac{3}{n} \right]
\]

\[
= \lim_{n \to \infty} \frac{3}{n} \sum_{i=1}^{n} \left[ \frac{9}{n^2} i^2 + 1 \right]
\]

\[
= \lim_{n \to \infty} \left[ \frac{27}{n^3} \sum_{i=1}^{n} i^2 + \frac{3}{n} \sum_{i=1}^{n} 1 \right]
\]

\[
= \lim_{n \to \infty} \left[ \frac{27}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} + \frac{3}{n} \cdot n \right]
\]

\[
= \lim_{n \to \infty} \left[ \frac{9(2n^2 + 3n + 1)}{2n^2} + 3 \right]
\]

\[
= 12
\]
Example

Use midpoint evaluation to approximate the definite integral

$$\int_{0}^{3} \sqrt{x^2 + 1} \, dx.$$
Example

Use midpoint evaluation to approximate the definite integral

\[ \int_{0}^{3} \sqrt{x^2 + 1} \, dx. \]

\[ \int_{0}^{3} \sqrt{x^2 + 1} \, dx \approx \sum_{i=1}^{n} \left[ \sqrt{\left( \left( \frac{i}{n} - \frac{1}{2} \right) \cdot \frac{3}{n} \right)^2 + 1} \cdot \frac{3}{n} \right] \]

Since we do not have a summation formula for square roots we cannot take the limit as \( n \to \infty \). Instead we will experiment with some “large” values of \( n \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>( A_n )</th>
</tr>
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<tbody>
<tr>
<td>10</td>
<td>5.6491</td>
</tr>
<tr>
<td>50</td>
<td>5.6525</td>
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<tr>
<td>100</td>
<td>5.6526</td>
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Signed Area

**Definition**
Suppose that $f(x) \geq 0$ on the interval $[a, b]$ and $A_1$ is the area bounded between $y = f(x)$ and the $x$-axis for $a \leq x \leq b$. Suppose that $f(x) \leq 0$ on the interval $[b, c]$ and $A_2$ is the area bounded between $y = f(x)$ and the $x$-axis for $b \leq x \leq c$. The **signed area** between $y = f(x)$ and the $x$-axis for $a \leq x \leq c$ is $A_1 - A_2$. The **total area** between $y = f(x)$ and the $x$-axis for $a \leq x \leq c$ is $A_1 + A_2$.

**Remark:** the definite integral can be interpreted as signed area.
Illustration

\[ \int_a^c f(x) \, dx = A_1 - A_2 \]
Express as a definite integral the total area between $y = 6x - x^2$ and the $x$-axis on the interval $[0, 6]$. 
Graph

\[ A = \int_0^6 (6x - x^2) \, dx \]
Example

Express as a definite integral or sum of definite integrals the total area between \( y = x^3 - 4x \) and the \( x \)-axis on the interval \([-2, 3]\).
$A = \int_{-2}^{0} (x^3 - 4x) \, dx - \int_{0}^{2} (x^3 - 4x) \, dx + \int_{2}^{3} (x^3 - 4x) \, dx$
Example

Suppose \( f(x) = \begin{cases} 
-x & \text{if } x < 0 \\
\sqrt{4 - x^2} & \text{if } 0 \leq x \leq 2 
\end{cases} \) and evaluate

\[
\int_{-2}^{2} f(x) \, dx.
\]
Example

Suppose \( f(x) = \begin{cases} 
-x & \text{if } x < 0 \\
\frac{-x}{\sqrt{4-x^2}} & \text{if } 0 \leq x \leq 2
\end{cases} \) and evaluate

\[
\int_{-2}^{2} f(x) \, dx.
\]

\[
\int_{-2}^{2} f(x) \, dx = \frac{1}{2}(2)(2) + \frac{1}{4}\pi(2)^2 = 2 + \pi
\]
Application

An object moving along a straight line has velocity function $v(t) = \cos t$. If the object starts at position 0,

1. determine the total distance the object travels between $t = 0$ and $t = 3\pi/2$, and

2. determine the net distance the object travels between $t = 0$ and $t = 3\pi/2$. 
Solution

1. Total distance the object travels between $t = 0$ and $t = \frac{3\pi}{2}$.

$$T = \int_{0}^{\pi/2} \cos t \, dt - \int_{\pi/2}^{3\pi/2} \cos t \, dt$$
1. Total distance the object travels between $t = 0$ and $t = \frac{3\pi}{2}$.

\[ T = \int_0^{\pi/2} \cos t \, dt - \int_{\pi/2}^{3\pi/2} \cos t \, dt \]

2. Net distance the object travels between $t = 0$ and $t = \frac{3\pi}{2}$.

\[ N = \int_0^{3\pi/2} \cos t \, dt \]
Theorem

If $f$ is continuous on $[a, b]$ then $f$ is integrable on $[a, b]$.

Remark: in fact $f$ remains integrable even when it has a finite number of jump or removable discontinuities.
Properties of the Definite Integral (2 of 3)

Theorem

If $f$ and $g$ are integrable on $[a, b]$, then

1. For any constants $c$ and $d$,

\[
\int_{a}^{b} (c f(x) + d g(x)) \, dx = c \int_{a}^{b} f(x) \, dx + d \int_{a}^{b} g(x) \, dx,
\]

2. For any $c$ in interval $[a, b]$,

\[
\int_{a}^{b} f(x) \, dx = \int_{a}^{c} f(x) \, dx + \int_{c}^{b} f(x) \, dx.
\]
The following properties also hold for definite integrals:

\[
\int_{a}^{b} f(x) \, dx = - \int_{b}^{a} f(x) \, dx
\]

\[
\int_{a}^{a} f(x) \, dx = 0 \quad \text{if } f(a) \text{ is defined.}
\]
Example

Write the expression

\[ \int_{-2}^{1} f(x) \, dx - \int_{-1}^{1} f(x) \, dx \]

as a single integral.
Example

Write the expression

\[ \int_{-2}^{1} f(x) \, dx - \int_{-1}^{1} f(x) \, dx \]

as a single integral.

\[ \int_{-2}^{1} f(x) \, dx - \int_{-1}^{1} f(x) \, dx = \int_{-2}^{1} f(x) \, dx + \int_{1}^{-1} f(x) \, dx \]

= \int_{-2}^{-1} f(x) \, dx
Example

If \( \int_{-1}^{3} f(x) \, dx = 8 \) and \( \int_{-1}^{3} g(x) \, dx = -2 \) find the value of

\[
\int_{-1}^{3} \left( \frac{1}{3} f(x) + 2g(x) \right) \, dx.
\]
Example

If \( \int_{-1}^{3} f(x) \, dx = 8 \) and \( \int_{-1}^{3} g(x) \, dx = -2 \) find the value of

\[
\int_{-1}^{3} \left( \frac{1}{3} f(x) + 2g(x) \right) \, dx.
\]

\[
\int_{-1}^{3} \left( \frac{1}{3} f(x) + 2g(x) \right) \, dx = \frac{1}{3} \int_{-1}^{3} f(x) \, dx + 2 \int_{-1}^{3} g(x) \, dx
\]

\[
= \frac{1}{3} (8) + 2(-2) = \frac{8}{3} - 4 = -\frac{4}{3}
\]
Definite Integral Inequality

Theorem

Suppose that \( g(x) \leq f(x) \) for \( a \leq x \leq b \) and that \( f \) and \( g \) are integrable on \([a, b]\). Then

\[
\int_a^b g(x) \, dx \leq \int_a^b f(x) \, dx.
\]

Proof.

\[
0 \leq f(x) - g(x)
\]

\[
\int_a^b 0 \, dx \leq \int_a^b (f(x) - g(x)) \, dx
\]

\[
0 \leq \int_a^b f(x) \, dx - \int_a^b g(x) \, dx
\]
Average Value of a Function

Suppose $f$ is continuous on $[a, b]$ and $\{x_0, x_1, \ldots, x_n\}$ is a regular partition of $[a, b]$, then the average of $\{f(x_1), f(x_2), \ldots, f(x_n)\}$ is

$$\frac{1}{n} \sum_{i=1}^{n} f(x_i) = \frac{1}{n} \sum_{i=1}^{n} f(x_i) \frac{b - a}{n} \frac{n}{b - a}$$

$$= \frac{1}{b - a} \sum_{i=1}^{n} f(x_i) \Delta x$$

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} f(x_i) = \lim_{n \to \infty} \frac{1}{b - a} \sum_{i=1}^{n} f(x_i) \Delta x$$

$$f_{avg} = \frac{1}{b - a} \int_{a}^{b} f(x) \, dx.$$
Example

Estimate the average value of $\cos x$ on the interval $[0, \pi/2]$. 

Graph showing the function $y = \cos x$ on the interval $[0, \pi/2]$, with the average value approximated by a horizontal line at $y = 0.6$. Key points marked at $\pi/8$, $\pi/4$, $3\pi/8$, and $\pi/2$. 
Solution

\[ f_{\text{avg}} = \frac{1}{\frac{\pi}{2} - 0} \int_{0}^{\frac{\pi}{2}} \cos x \, dx \]

Use a Riemann sum with \( n = 100 \) and midpoint evaluation to approximate the definite integral.

\[ \Delta x = \frac{\frac{\pi}{2} - 0}{100} = \frac{\pi}{200} \]

\[ c_i = 0 + \left( i - \frac{1}{2} \right) \frac{\pi}{200} = \left( i - \frac{1}{2} \right) \frac{\pi}{200} \]

\[ \int_{0}^{\frac{\pi}{2}} \cos x \, dx \approx \sum_{i=1}^{100} \cos \left( \left( i - \frac{1}{2} \right) \frac{\pi}{200} \right) \frac{\pi}{200} = 1.00001 \]

\[ f_{\text{avg}} \approx \frac{2}{\pi} (1.00001) = 0.636626 \]
**Integral Mean Value Theorem**

**Theorem (Integral Mean Value Theorem)**

*If $f$ is continuous on $[a, b]$ there exists $a \leq c \leq b$ such that*

\[ f(c) = \frac{1}{b-a} \int_{a}^{b} f(x) \, dx. \]
Proof

By the Extreme Value Theorem $f$ has an absolute minimum $m$ and an absolute maximum $M$ on $[a, b]$.

\[
m \leq f(x) \leq M
\]

\[
\int_a^b m \, dx \leq \int_a^b f(x) \, dx \leq \int_a^b M \, dx
\]

\[
m(b - a) \leq \int_a^b f(x) \, dx \leq M(b - a)
\]

\[
m \leq \frac{1}{b - a} \int_a^b f(x) \, dx \leq M
\]

By the Intermediate Value Theorem the result follows.
Example

Suppose \( \int_{-1}^{1} (x^2 - 2x) \, dx = \frac{2}{3} \) and find the value of \( c \) that satisfies the conclusion of the Integral Mean Value Theorem.
Example

Suppose \( \int_{-1}^{1} (x^2 - 2x) \, dx = \frac{2}{3} \) and find the value of \( c \) that satisfies the conclusion of the Integral Mean Value Theorem.

\[
\begin{align*}
    f(x) &= \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \\
    c^2 - 2c &= \frac{1}{1-(-1)} \int_{-1}^{1} (x^2 - 2x) \, dx \\
    c^2 - 2c &= \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3} \\
    3c^2 - 6c - 1 &= 0 \\
    c &= \frac{6 \pm \sqrt{36 - 4(3)(-1)}}{6} = 1 \pm \frac{2\sqrt{3}}{3}
\end{align*}
\]

Note \( c = 1 - \frac{2\sqrt{3}}{3} \approx -0.154701 \) lies in the interval \([-1, 1]\).
Homework

- Read Section 4.4
- Exercises: 1, 5, 9, 13, 17, 21, 25, 29, 33, 35, 37, 39, 45, 47, 49, 51