Maximum and Minimum Values
MATH 161 Calculus I

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In almost all quantitative fields there are objective outcomes that we would like to optimize (either by maximizing or minimizing some quantity).
Why Maximize or Minimize?

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▶ A business owner wants to maximize revenue while minimizing cost.

▶ An investor wishes to maximize return on investment while minimizing risk of loss of investment.

▶ An electrical engineer must minimize power consumption by an electrical device.
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Absolute Extrema

Definition
Let $f$ be a function defined on a set $S$ of real numbers and let $c \in S$.

1. $f(c)$ is the **absolute maximum** of $f$ on $S$ if $f(c) \geq f(x)$ for all $x \in S$.

2. $f(c)$ is the **absolute minimum** of $f$ on $S$ if $f(c) \leq f(x)$ for all $x \in S$.

An absolute maximum or an absolute minimum is referred to as an **absolute extremum**. If a function has more than one extremum, we refer to these as **extrema**.
Example (1 of 4)

Suppose \( f(x) = \frac{x^2}{(x-1)^2} \) and \( S = (-\infty, 1) \cup (1, \infty) \). Does \( f \) have an absolute maximum and absolute minimum on \( S \)?
Example (2 of 4)

Suppose \( f(x) = \frac{x^2}{(x - 1)^2} \) and \( S = [-1, 1) \). Does \( f \) have an absolute maximum and absolute minimum on \( S \)?
Example (3 of 4)

Suppose \( f(x) = \frac{x^2}{(x - 1)^2} \) and \( S = [-1, -1/2] \). Does \( f \) have an absolute maximum and absolute minimum on \( S \)?
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Extreme Value Theorem

**Question:** under what conditions will a function have an absolute maximum and absolute minimum?
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Theorem (Extreme Value Theorem)

A continuous function $f$ defined on a closed, bounded interval $[a, b]$ attains both an absolute maximum and an absolute minimum on that interval.
Example

Find the absolute extrema of $f(x) = x^3 - 3x^2 + 3x + 1$ on the interval $[0, 2]$. 

![Graph showing absolute extrema]
Other Types of Extrema

We would like to reliably find the absolute extrema of a function. In order to develop a procedure for this we should understand there are other types of extrema.

Definition

\[ f(c) \] is a local maximum of \( f \) if \( f(c) \geq f(x) \) for all \( x \) in some open interval containing \( c \).

\[ f(c) \] is a local minimum of \( f \) if \( f(c) \leq f(x) \) for all \( x \) in some open interval containing \( c \).

In either case, we call \( f(c) \) a local extremum.

The terms relative maximum, relative minimum, and relative extremum are used in some textbooks.
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Where are Local Extrema Found?

Consider the graph below and determine the points where an extremum may occur.
It seems that an extremum can occur where $f'(x) = 0$ or $f'(x)$ is undefined.
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Local vs. Absolute Extrema

- Local min.
- Local max.
- Abs. min.
- Abs. max.

The graph shows a function $f(x)$ with points indicating local maxima and minima as well as absolute maxima and minima.
Critical Numbers

Definition
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**Recall**: features in graphs such as vertical tangents, corners, or cusps indicate that $f'(c)$ is undefined.
Fermat’s Theorem

Theorem (Fermat’s Theorem)

Suppose that \( f(c) \) is a local extremum, then \( c \) is a critical number of \( f \).
Proof

Suppose that $f(c)$ is a local extremum.

- If $f$ is not differentiable at $c$, then by definition $c$ is a critical number.

- Suppose $f$ is differentiable at $c$ and suppose $f'(c) > 0$, then $f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} > 0 \implies f(c+h) - f(c) > 0$ when $h$ is small.

- If $h > 0$ this implies $f(c+h) > f(c)$, so $f(c)$ is not a maximum.

- If $h < 0$ this implies $f(c+h) < f(c)$, so $f(c)$ is not a minimum.

This contradiction implies $f'(c) \neq 0$. Similarly we may show $f'(c) \neq 0$ and thus $f'(c) = 0$. 
Proof

Suppose that \( f(c) \) is a local extremum.

- If \( f \) is not differentiable at \( c \), then by definition \( c \) is a critical number.
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f'(c) = \lim_{h \to 0} \frac{f(c + h) - f(c)}{h} > 0 \quad \Rightarrow \quad \frac{f(c + h) - f(c)}{h} > 0
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- If \( h > 0 \) this implies \( f(c + h) > f(c) \), so \( f(c) \) is not a maximum.
- If \( h < 0 \) this implies \( f(c + h) < f(c) \), so \( f(c) \) is not a minimum.

This contradiction implies \( f'(c) \neq 0 \). Similarly we may show \( f'(c) \neq 0 \) and thus \( f'(c) = 0 \).
Examples

Find the critical numbers of the following functions.

- \( f(x) = x^4 + 6x^2 - 2 \)
- \( g(x) = \frac{x^2 - x + 4}{x - 1} \)
- \( h(x) = x^{7/3} - 28x^{1/3} \)
Solutions

▶ $f(x) = x^4 + 6x^2 - 2$
Solutions

- \( f(x) = x^4 + 6x^2 - 2 \)

\[ f'(x) = 4x(x^2 + 3) \]

\( x = 0 \) is the only critical number.

- \( g(x) = x^2 - x + 4 \)

\[ g'(x) = x^2 - 2x - 3 = (x+1)(x-3)^2 \]

\( x = -1 \) and \( x = 3 \) are critical numbers. Why is \( x = 1 \) not a critical number?

- \( h(x) = x^{7/3} - 28x^{1/3} \)

\[ h'(x) = \frac{7}{3}x^{2/3} - \frac{4}{3}x^{-2/3} = \frac{7}{3}x^{2/3}(x-2) \]

\( x = -2, \ x = 0, \ x = 2 \) are critical numbers.
Solutions

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- \( g(x) = \frac{x^2 - x + 4}{x - 1} \)

\[ g'(x) = \frac{(x^2 - x + 4)(x - 1) - (x^2 - x + 4)}{(x - 1)^2} = \frac{7x^2 - 4x + 7}{(x - 1)^2} \]

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Solutions

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h'(x) = \frac{7}{3} \frac{x^2 - 4}{x^{2/3}} = \frac{7}{3} \frac{(x - 2)(x + 2)}{x^{2/3}}
\]
\( x = -2, x = 0, \) and \( x = 2 \) are critical numbers.
Main Result

Theorem
Suppose that $f$ is continuous on a closed, bounded interval $[a, b]$. The absolute extrema of $f$ must occur at the endpoints of the interval ($x = a$ or $x = b$) or at a critical number.
Procedure for Finding Absolute Extrema

Given a continuous function $f$ on a closed, bounded interval $[a, b]$:

1. Find all critical numbers in the interval and compute function values at these points.
2. Compute function values at the endpoints.
3. The largest function value is the absolute maximum and the smallest value found is the absolute minimum.
Example

Find the absolute extrema of \( f(x) = x^4 - 8x^2 + 2 \) on the interval \([-1, 3]\).
Solution

\[ f(x) = x^4 - 8x^2 + 2 \]
\[ f'(x) = 4x(x^2 - 4) \]
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The critical points are \( c = 0 \) and \( c = 2 \) (note \( x = -2 \) is not in the interval \([-1, 3]\)).
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\[ f(-1) = -5 \]
\[ f(0) = 2 \]
\[ f(2) = -14 \text{ (absolute minimum)} \]
\[ f(3) = 11 \text{ (absolute maximum)} \]
Illustration

The graph shows a function $f(x)$ plotted against $x$.

Key points on the graph include:
- $x = -1$, $f(x) = -5$
- $x = 1$, $f(x) = 5$
- $x = 3$, $f(x) = 10$

The function appears to be a parabola opening upwards.
Example

Find the absolute extrema of $f(x) = \sin x + \cos x$ on the interval $[\pi/2, 3\pi/2]$. 
Solution

\[ f(x) = \sin x + \cos x \]
\[ f'(x) = \cos x - \sin x \]
Solution

\[ f(x) = \sin x + \cos x \]
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The only critical point in the interval \([\pi/2, 3\pi/2]\) is \( c = 5\pi/4 \).
Solution

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\[ f'(x) = \cos x - \sin x \]
\[ 0 = \cos x - \sin x \]
\[ \sin x = \cos x \]

The only critical point in the interval \([\pi/2, 3\pi/2]\) is \(c = 5\pi/4\).

\[ f \left( \frac{\pi}{2} \right) = 1 \text{ (absolute maximum)} \]
\[ f \left( \frac{5\pi}{4} \right) = -\sqrt{2} \text{ (absolute minimum)} \]
\[ f \left( \frac{3\pi}{2} \right) = -1 \]
Homework

- Read Section 3.3
- Exercises: 1–47 odd