Motivation

We now return to one of the most important applications of the derivative: finding the maximum or minimum of a function.
Guidelines

- Draw a picture, if applicable.
- Determine the variables of the problem and how they are related.
- Decide which quantity should be optimized.
- Find an expression for the quantity to be optimized. This expression should contain only one variable.
- Determine the minimum and maximum allowable values (if any) for the variable.
- Apply the First or Second Derivative Test to optimize the quantity.
Example

Farmer MacDonald has 300 feet of chicken wire with which to construct a rectangular pen to hold a flock of chickens. A 400-ft–long chicken coop will be used to form one side of the pen, so the wire is needed for the remaining three sides. How can the pen be constructed so that the birds have the maximum space in which to roam?
Example

A rectangle is to be inscribed in a semicircle with radius 2 inches. Find the dimensions of the rectangle that encloses the maximum area.
Solution

- If the center of the base of the rectangle is at $x = 0$ and the center of the diameter of the semicircle is also at $x = 0$, then the equation of the semicircle is $y = \sqrt{4 - x^2}$.

- Suppose the endpoints of the base of the rectangle are at $(-w, 0)$ and $(w, 0)$ (with $0 \leq w \leq 2$), then the height of the rectangle is $h = \sqrt{4 - w^2}$. Thus the area of the rectangle is

\[
A(w) = 2w \sqrt{4 - w^2}
\]
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$$A'(w) = \frac{8 - 4w^2}{\sqrt{4 - w^2}}$$
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w = \sqrt{2}
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  \]

  \[
  w = \sqrt{2}
  \]

- Using Fermat’s theorem we can check the absolute maximum area occurs when $w = \sqrt{2}$.

  \[
  A(0) = 0, \quad A(2) = 0, \quad A(\sqrt{2}) = 4
  \]

  The dimensions of the rectangle are $2\sqrt{2}$ in. by $\sqrt{2}$ in.
Example

Find the point on the graph of $f(x) = x^2$ that is closest to the point with coordinates $(3, 4)$. 
The distance from the point \((x, x^2)\) to \((3, 4)\) is
\[
\sqrt{(x - 3)^2 + (x^2 - 4)^2}.
\]

If we minimize the expression under the square root we will minimize the square root as well. Thus let
\[
f(x) = (x - 3)^2 + (x^2 - 4)^2 = x^4 - 7x^2 - 6x + 25.
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Find the critical numbers of \(f\).

\[
f'(x) = 4x^3 - 14x - 6
\]

We can plot \(f'(x)\) and estimate the critical numbers using Newton’s Method.
According to Newton’s Method the positive critical number is \( c \approx 2.05655 \).
Using the Second Derivative Test we see that

\[ f''(x) = 12x^2 - 14 \]

\[ f''(2.05655) = 36.7525 > 0 \]

and thus the minimum distance between the parabola and the point with coordinates \((3, 4)\) occurs at

\[ (2.05655, (2.05655)^2) \approx (2.05655, 4.22938). \]
Example

Find the ratio of the radius to the height of a right circular cylinder that has minimal surface area, assuming that the volume of the cylinder is constant and the cylinder has a top and a bottom.
Example

A beer distributor has determined that 200 gallons of lite beer can be sold to a tavern if the price is $2 per gallon. For each cent per gallon that the price is lowered 2.5 more gallons will be sold. At what price should the beer be sold in order to maximize the revenue to the distributor?
The previous example would be more realistic if instead of maximizing revenue received, we could determine the price that would maximize the distributor’s profit. Suppose the distributor must pay $0.90 per gallon to the brewery. What price would maximize the profit, assuming the other conditions remain the same?
Example

A company needs to run an oil pipeline from an oil rig 25 miles out at sea to a storage tank that is 5 miles inland. The shoreline runs east-west and the tank is 8 miles east of the rig. It costs $60,000 per mile to construct the pipeline under water and $25,000 per mile to construct the pipeline on land. The pipeline will be built in a straight line from the rig to the shore and then in a straight line to the storage tank. What point on the shoreline should be selected to minimize the total cost of the pipeline?
Homework

- Read Section 3.7
- Exercises: 1–21 odd