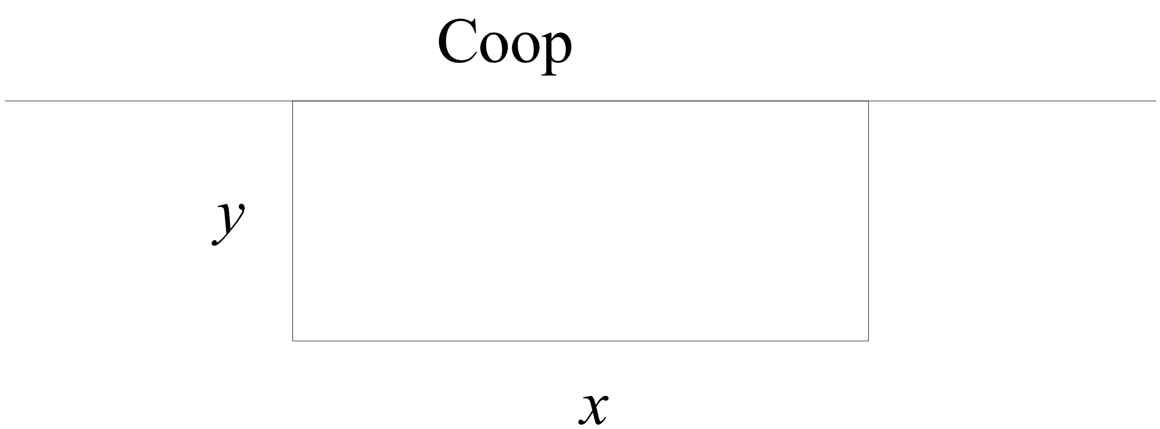
Example Optimization Problems

1. A farmer has 300 feet of chicken wire with which to construct a rectangular pen to hold a flock of chickens. A 400-foot long chicken coop will be used for one side of the pen, so the wire is only needed for the remaining three sides of the pen. How can the pen be constructed so that the chickens have the maximum space in which to roam?

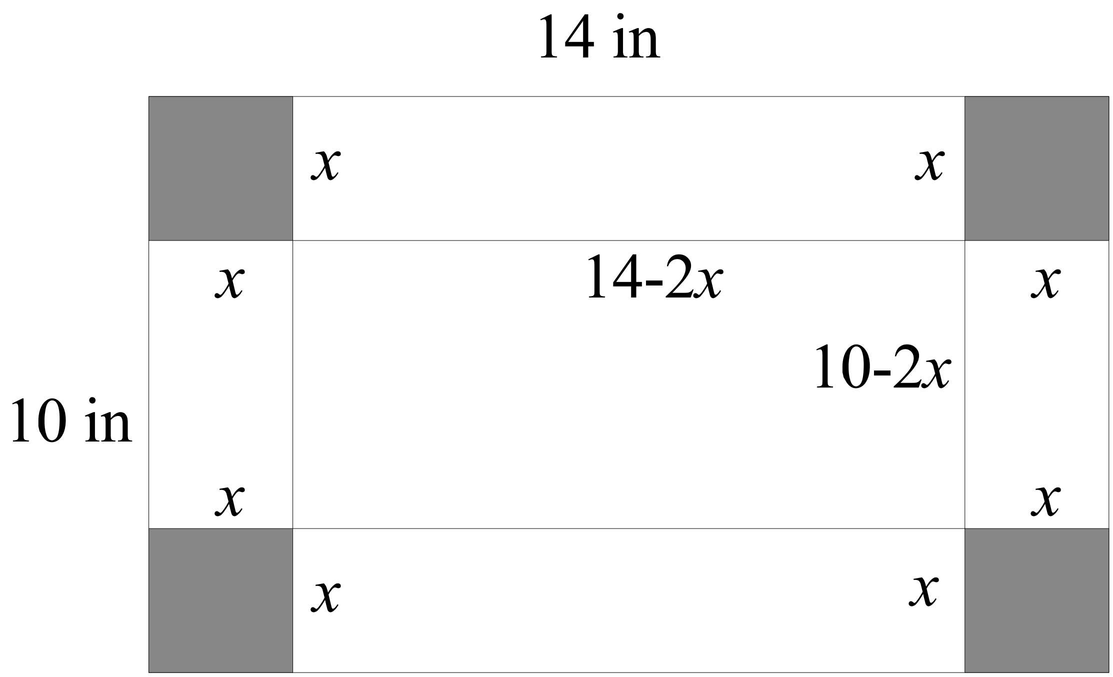


If we let the base of the pen have length x and the height be y then the area of the pen is $A = xy$. The dimensions of the pen are constrained so that $300 = x + 2y$. Solving the constraint for y we see that $y = 150 - x/2$. Thus we can derive a formula for the area of the pen which only depends on x, namely

$$A(x) = x(150 - \frac{x}{2}) = 150x - \frac{x^2}{2}.$$

We wish to maximize this function. $A'(x) = 150 - x$ and thus the only critical number is $x = 150$. We know the global maximum occurs at $x = 150$ and $y = 150 - 150/2 = 75$ since the area function produces a graph which is a downward opening parabola.

2. An open box will be constructed from a single sheet of cardboard. The sheet is 10 inches wide and 14 inches long. A square will be cut from each corner and the resulting sides folded up to form a rectangular box with no top. What size square cut from each corner of the sheet will create the box with the largest volume?



If we let the length of an edge of the square removed be x then the dimensions of the resulting box are (length) $14-2x$, (width) $10-2x$, and (height) x. Thus the volume of the box is

$$V(x) = (14-2x)(10-2x)x = 4x^3 - 48x^2 + 140x.$$

We wish to maximize this function.

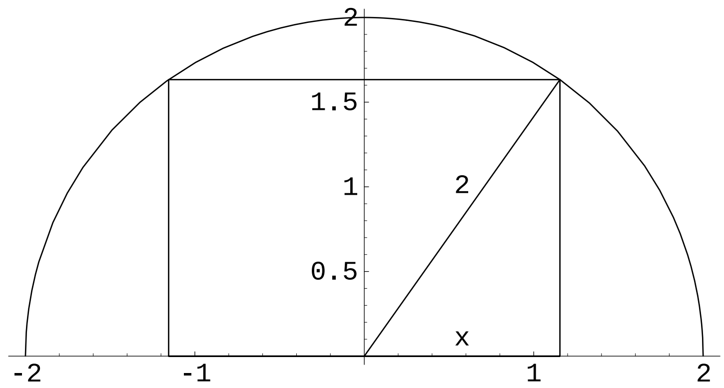
$$\begin{aligned} V'(x) &= 12x^2 - 96x + 140 \\ &= 4(3x^2 - 24x + 35) \end{aligned}$$

Setting the derivative equal to zero and using the quadratic formula produces two critical numbers

$$x = 4 + \frac{\sqrt{39}}{3} \approx 6.081666 \quad \text{and} \quad x = 4 - \frac{\sqrt{39}}{3} \approx 1.191833$$

Since $x \geq 5$ would make the width of the box negative we will ignore the larger of the two critical numbers. If we use the Extreme Value Theorem, then we see that the volume is maximized when (height) $x = 4 - \frac{\sqrt{39}}{3}$, (length) $14 - 2x \approx 10.16333$, and (width) $10 - 2x \approx 6.16333$.

3. A rectangle is to be inscribed in a semicircle of radius 2 inches. Find the dimensions of the rectangle that encloses the maximum area.



Using the dimensions labeled in the figure above the base of the rectangle has length $2x$ and its height is $\sqrt{4-x^2}$. Thus the area of the rectangle is given by the function, $A(x) = 2x\sqrt{4-x^2}$. We must find the critical points of $A(x)$.

$$\begin{aligned} A'(x) &= 2\sqrt{4-x^2} + 2x(4-x^2)^{-1/2}(-2x) \\ &= 2\sqrt{4-x^2} + \frac{-4x^2}{\sqrt{4-x^2}} \\ &= \frac{2(4-x^2)-4x^2}{\sqrt{4-x^2}} \\ &= \frac{2(4-3x^2)}{\sqrt{4-x^2}} \end{aligned}$$

Thus $A'(x) = 0$ when $4 - 3x^2 = 0$ which is true when $x = \pm\sqrt{4/3} = \pm 2/sqrt3$. We can ignore the negative solution. We also know that $0 \leq x \leq 2$ and thus according to the Extreme Value Theorem the maximum area occurs when $x = 2/\sqrt{3}$.

$$\begin{aligned} A(0) &= 0 \\ A(2/\sqrt{3}) &= 0 \\ A(2) &= 0 \end{aligned}$$

4. Find two positive real numbers whose product is 16 and whose sum is a minimum.

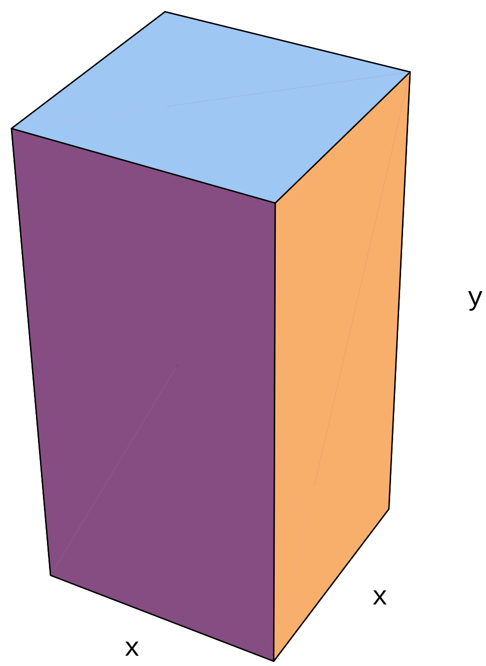
   Suppose the two numbers are x and y. We know that $x, y > 0$ and $xy = 16$. The last equation is a constraint on the choices for x and y. The sum of x and y is $S = x + y$. Solving for y in the constraint equation we can write S as a function of x only, $S(x) = x + 16/x$. To minimize $S(x)$ we look for the critical points.

$$\begin{aligned} S'(x) &= 1 - \frac{16}{x^2} \\ 0 &= 1 - \frac{16}{x^2} \\ \frac{16}{x^2} &= 1 \end{aligned}$$

$$
\begin{aligned}
16 &= x^2 \\
\pm 4 &= x
\end{aligned}
$$

We can ignore the solution $x = -4$ since x must be positive. Since $S'(x) < 0$ on the interval $(0, 4)$ and $S'(x) > 0$ on $(4, \infty)$ then the absolute minimum occurs when $x = y = 4$.

5. A rectangular box with a square top and bottom is to have a volume of 2250 cm^3. Find the dimensions of the box with the minimum cost if material for the top and bottom of the box costs \$0.04 per square cm and material for the sides costs \$0.015 per square cm.



Let the dimensions of the base be x by x (since the base is a square) and the height of the box be y. The dimensions are constrained by the fact that the volume must be $2250 = x^2 y$. The cost of constructing the box will be the sum of the costs of the six sides of the box. The cost of the side of a box will be its area multiplied by the cost of the material per unit area. Let the cost be

$$C = 2(0.04x^2) + 4(0.015xy) = 0.08x^2 + 0.06xy.$$

Solving the constraint equation for y yields $y = 2250/x^2$ and thus we can write the cost as a function of x only,

$$C(x) = 0.08x^2 + 0.06x(2250/x^2) = 0.08x^2 + \frac{135}{x}.$$

We can minimize the cost by looking for the critical points of the function $C(x)$.

$$\begin{aligned} C'(x) &= 0.16x - \frac{135}{x^2} \\ 0 &= 0.16x - \frac{135}{x^2} \\ \frac{135}{x^2} &= 0.16x \\ \frac{135}{0.16} &= x^3 \\ 843.75 &= x^3 \\ \sqrt[3]{843.75} &= x \\ x &\approx 9.44941 \end{aligned}$$

We can see from the first derivative that $C'(x) < 0$ on the interval $(0, 9.44941)$ and $C'(x) > 0$ for $(9.44941, \infty)$. Thus the absolute minimum of cost occurs when $x = 9.44941$. Thus the dimensions of the minimum cost box are (length) 9.44941 cm, (width) 9.44941 cm, and (height) 25.1984 cm.

6. A standard can has a volume of 900 cm^3. The can is in the shape of a right circular cylinder with a top and a bottom. Find the dimensions of the can that minimize the amount of material needed for construction.

   Suppose that the cylinder has radius r and height h. The volume of a cylinder is given by the formula $V = \pi r^2 h$. Thus the dimensions of the cylinder are constrained by the equation $900 = \pi r^2 h$. The material needed to construct the can is minimized when the total surface area of the can is minimized. The surface area includes the top, bottom, and the curved side. We will let the surface area be S, where

$$S = 2(\pi r^2) + 2\pi r h.$$

   Solving the constraint equation for $h = 900/(\pi r^2)$ allows us to write the S as a function of r only.

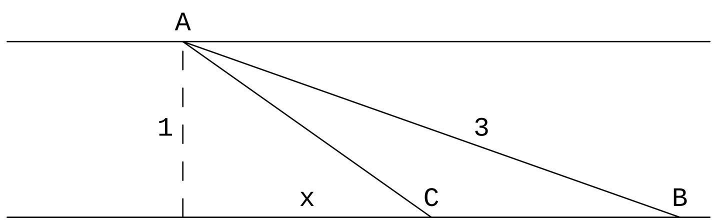
$$S(r) = 2\pi r^2 + 2\pi r \cdot \frac{900}{\pi r^2} = 2\pi r^2 + \frac{1800}{r}$$

   Thus we can minimize the surface area by finding the critical points of $S(r)$.

$$\begin{aligned} S'(r) &= 4\pi r - \frac{1800}{r^2} \\ 0 &= 4\pi r - \frac{1800}{r^2} \\ \frac{1800}{r^2} &= 4\pi r \\ \frac{1800}{4\pi} &= r^3 \\ \sqrt[3]{\frac{450}{\pi}} &= r \end{aligned}$$

Thus the critical point occurs at $r \approx 5.23224$. We can see from a sign diagram of the first derivative that $S'(r) < 0$ for r in the interval $(0, \sqrt[3]{450/\pi})$ and $S'(r) > 0$ on the interval $(\sqrt[3]{450/\pi}, \infty)$. Thus the global minimum occurs when $r = \sqrt[3]{450/\pi}$. Substituting this value in the constraint equation implies that $h = 2\sqrt[3]{450.0/Pi} \approx 10.4645$.

7. A pipeline for transporting oil will connect two points A and B that are 3 miles apart and on opposite sides of a river that is one mile wide. Part of the pipeline will run under water from A to a point C on the opposite bank and then above ground from C to B. If the cost of running the pipeline under water is four times the cost of running it above ground, find the location of C that will minimize the total cost of the pipeline.



Using the Pythagorean Theorem we see that the distance from the point directly across the river from point A to point B is $\sqrt{3^2 - 1} = \sqrt{8} = 2\sqrt{2}$. The distance from the point directly across the river from point A to point C is labeled x. The total cost of the pipeline will be the sum of cost of the underwater portion and the cost of the above ground portion. We can write the total cost as a function of x,

$$C(x) = 4\sqrt{1 + x^2} + 2\sqrt{2} - x.$$

The cost is minimized at a critical point of $C(x)$.

$$\begin{aligned}
C'(x) &= 4\frac{1}{2}(1 + x^2)^{-1/2}2x - 1 \\
&= \frac{4x}{\sqrt{1 + x^2}} - 1 \\
0 &= \frac{4x}{\sqrt{1 + x^2}} - 1 \\
1 &= \frac{4x}{\sqrt{1 + x^2}} \\
\sqrt{1 + x^2} &= 4x \\
1 + x^2 &= 16x^2 \\
1 &= 15x^2 \\
\pm\sqrt{\frac{1}{15}} &= x
\end{aligned}$$

The cost cannot be minimized when $x < 0$ since the underwater pipeline would be heading away from point B. The value of x which minimizes the cost must lie in the closed interval $[0, 2\sqrt{2}]$. According to the Extreme Value Theorem the cost is minimized when $x = \sqrt{1/15} \approx 0.258199$, since

$$\begin{aligned} C(0) &= 4 + 2\sqrt{2} \approx 6.82843 \\ C(\sqrt{1/15}) &= 2\sqrt{2} + \sqrt{15} \approx 6.70141 \\ C(2\sqrt{2}) &= 12 \end{aligned}$$

8. A beer distributor has determined that 200 gallons of light beer can be sold to a tavern if the price is \$2 per gallon. The beer costs the distributor \$0.90 per gallon. For each cent that the price to the tavern is lowered, 10 more quarts of beer will be sold. At what price should the beer be sold to the tavern so that the distributor has the greatest profit?

   Profit is revenue minus cost. Revenue is the number of gallons of beer sold times the price per gallon to the tavern's customers. Cost is the number of gallons sold times \$0.90, which is the cost per gallon to the tavern. If we create a small table we can discover a relationship between the customer price and the number of gallons sold.

Price per gallon	**Gallons sold**
2.00	200.0
1.99	202.5
1.98	205.0
1.97	207.5
⋮	⋮

Thus if the price of the beer is $2 - 0.01x$ dollars per gallon then $200 + 2.5x$ gallons will be bought. Hence we can write the profit P, as a function of x.

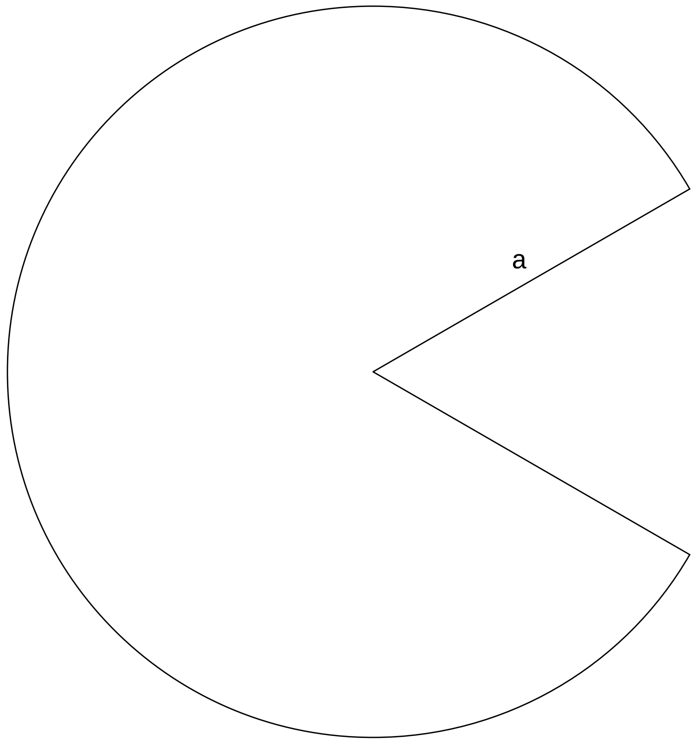
$$\begin{aligned} P(x) &= \text{revenue} - \text{cost} \\ &= (2 - 0.01x)(200 + 2.5x) - 0.90(200 + 2.5x) \\ &= (2 - 0.01x - 0.90)(200 + 2.5x) \\ &= (1.10 - 0.01x)(200 + 2.5x) \\ &= -0.025x^2 + 0.75x + 220 \end{aligned}$$

The graph of the profit will be a downward opening parabola. Thus the maximum value of the profit occurs at the vertex, which is occurs at the value of x for which the tangent line is horizontal.

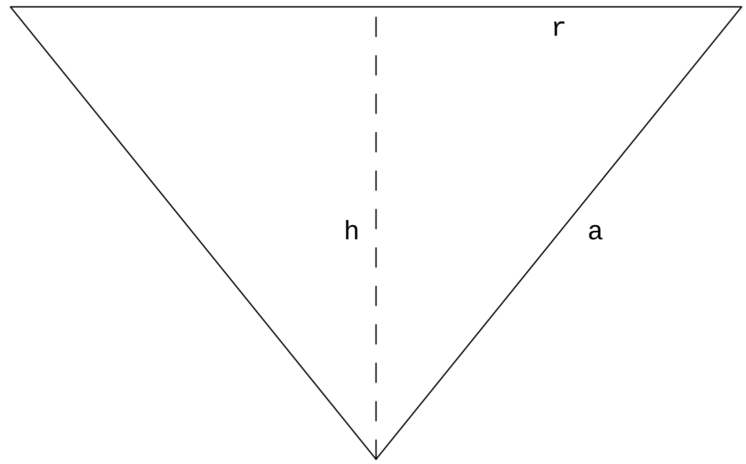
$$\begin{aligned} P'(x) &= -0.05x + 0.75 \\ 0 &= -0.05x + 0.75 \\ x &= 15 \end{aligned}$$

Thus the profit is maximized when the selling price of the beer is \$1.85 per gallon.

9. A paper drinking cup in the shape of a cone is to be constructed by removing a circular sector from a circular disk of paper of radius a and then joining the two straight edges. What is the largest cup volume?



We will assume the angle subtended by the sector is θ where $0 \leq \theta \leq 2\pi$. When the sector is removed the circle loses $a\theta$ of its circumference. Thus when the cone is formed the open end has a circumference of $2\pi a - a\theta = a(2\pi - \theta)$. Thus the radius of the cone is $r = \frac{1}{2\pi}a(2\pi - \theta) = a(1 - \frac{\theta}{2\pi})$. Consider the side view of the cone in the figure below.



Using the Pythagorean Theorem we can find the height of the cone.

$$\begin{aligned} h &= \sqrt{a^2 - r^2} \\ &= \sqrt{a^2 - a^2(1 - \frac{\theta}{2\pi})^2} \\ &= a\sqrt{1 - (1 - \frac{\theta}{2\pi})^2} \\ &= a\sqrt{1 - 1 + \frac{\theta}{\pi} - \frac{\theta^2}{4\pi^2}} \\ &= a\sqrt{\frac{\theta}{\pi} - \frac{\theta^2}{4\pi^2}} \\ &= a\sqrt{\frac{4\pi\theta - \theta^2}{4\pi^2}} \\ &= \frac{a}{2\pi}\sqrt{4\pi\theta - \theta^2} \end{aligned}$$

The volume of a cone is given by the formula $V = \frac{1}{3}\pi r^2 h$. Hence we can write the volume of our cone as a function of θ.

$$\begin{aligned} V(\theta) &= \frac{1}{3}\pi r^2 h \\ &= \frac{1}{3}\pi \left(a(1 - \frac{\theta}{2\pi}) \right)^2 \frac{a}{2\pi}\sqrt{4\pi\theta - \theta^2} \\ &= \frac{a^3}{6}\left(1 - \frac{\theta}{2\pi}\right)^2 \sqrt{4\pi\theta - \theta^2} \end{aligned}$$

Now we can look for the critical points of the volume function.

$$V'(\theta) = \frac{a^3}{6}\left[2\left(1 - \frac{\theta}{2\pi}\right)\left(\frac{-1}{2\pi}\right)\sqrt{4\pi\theta - \theta^2} + \left(1 - \frac{\theta}{2\pi}\right)^2 \frac{1}{2}(4\pi\theta - \theta^2)^{-1/2}(4\pi - 2\theta) \right]$$

$$
\begin{aligned}
&= \frac{a^3}{6}\left(1-\frac{\theta}{2\pi}\right)\left[2\left(\frac{-1}{2\pi}\right)\sqrt{4\pi\theta-\theta^2}+\left(1-\frac{\theta}{2\pi}\right)\frac{1}{2}(4\pi\theta-\theta^2)^{-1/2}(4\pi-2\theta)\right] \\
&= \frac{a^3}{6}\left(1-\frac{\theta}{2\pi}\right)\left[\left(\frac{-1}{\pi}\right)\sqrt{4\pi\theta-\theta^2}+\left(1-\frac{\theta}{2\pi}\right)(4\pi\theta-\theta^2)^{-1/2}(2\pi-\theta)\right] \\
&= \frac{a^3}{6}\left(1-\frac{\theta}{2\pi}\right)\left[\left(\frac{-1}{\pi}\right)(4\pi\theta-\theta^2)^{1/2}+\frac{\left(1-\frac{\theta}{2\pi}\right)(2\pi-\theta)}{(4\pi\theta-\theta^2)^{1/2}}\right] \\
&= \frac{a^3}{6}\left(1-\frac{\theta}{2\pi}\right)\left[\frac{-(4\pi\theta-\theta^2)+\pi\left(1-\frac{\theta}{2\pi}\right)(2\pi-\theta)}{\pi(4\pi\theta-\theta^2)^{1/2}}\right] \\
&= \frac{a^3}{6}\left(1-\frac{\theta}{2\pi}\right)\left[\frac{\pi\left(1-\frac{\theta}{2\pi}\right)(2\pi-\theta)-(4\pi\theta-\theta^2)}{\pi(4\pi\theta-\theta^2)^{1/2}}\right]
\end{aligned}
$$

Thus the critical numbers are the values of θ for which $\theta - \frac{1}{2\pi} = 0$ which implies $\theta = 2\pi$ and the value of θ for which

$$
\begin{aligned}
\pi\left(1-\frac{\theta}{2\pi}\right)(2\pi-\theta)-(4\pi\theta-\theta^2) &= 0 \\
\pi\left(2\pi-2\theta+\frac{\theta^2}{2\pi}\right)-4\pi\theta+\theta^2 &= 0 \\
2\pi^2-2\pi\theta+\frac{\theta^2}{2}-4\pi\theta+\theta^2 &= 0 \\
\frac{3}{2}\theta^2-6\pi\theta+2\pi^2 &= 0 \\
3\theta^2-12\pi\theta+4\pi^2 &= 0 \\
\theta &= \frac{12\pi\pm\sqrt{144\pi^2-48\pi^2}}{6} \\
\theta &= 2\pi\left(1\pm\frac{\sqrt{6}}{3}\right)
\end{aligned}
$$

Since $0 \le \theta \le 2\pi$ then we can ignore the solution $2\pi\left(1+\frac{\sqrt{6}}{3}\right)$. According to the Extreme Value Theorem the maximum volume for the cone occurs when $\theta = 2\pi\left(1-\frac{\sqrt{6}}{3}\right) \approx 1.15299$ radians which is approximately $66.0612°$.

$$
\begin{aligned}
V(0) &= 0 && (1) \\
V(2\pi\left(1-\frac{\sqrt{6}}{3}\right) &= \frac{2a^3\pi}{9\sqrt{3}} && (2) \\
V(2\pi) &= 0 && (3)
\end{aligned}
$$