

Millersville University  
Department of Mathematics  
MATH 161, *Calculus I*, Implicit Differentiation

Examples

Find  $\frac{dy}{dx}$  for the following equations.

1.  $y = 2 \sin(\pi x - y)$

$$\begin{aligned}\frac{d}{dx}(y) &= \frac{d}{dx}(2 \sin(\pi x - y)) \\ \frac{dy}{dx} &= 2 \cos(\pi x - y) \cdot \frac{d}{dx}(\pi x - y) \\ &= 2 \cos(\pi x - y) \cdot \left(\pi - \frac{dy}{dx}\right) \\ &= 2\pi \cos(\pi x - y) - 2 \cos(\pi x - y) \cdot \frac{dy}{dx} \\ \frac{dy}{dx} + 2 \cos(\pi x - y) \cdot \frac{dy}{dx} &= 2\pi \cos(\pi x - y) \\ \frac{dy}{dx}(1 + 2 \cos(\pi x - y)) &= 2\pi \cos(\pi x - y) \\ \frac{dy}{dx} &= \frac{2\pi \cos(\pi x - y)}{1 + 2 \cos(\pi x - y)}\end{aligned}$$

2.  $x + \sin y = xy$

$$\begin{aligned}\frac{d}{dx}(x + \sin y) &= \frac{d}{dx}(xy) \\ 1 + \cos y \cdot \frac{dy}{dx} &= y + x \frac{dy}{dx} \\ \cos y \cdot \frac{dy}{dx} - x \frac{dy}{dx} &= y - 1 \\ (\cos y - x) \frac{dy}{dx} &= y - 1 \\ (x - \cos y) \frac{dy}{dx} &= 1 - y \\ \frac{dy}{dx} &= \frac{1 - y}{x - \cos y}\end{aligned}$$

3.  $x^3 - xy + y^3 = 1$

$$\frac{d}{dx}(x^3 - xy + y^3) = \frac{d}{dx}(1)$$

$$\begin{aligned}
3x^2 - \left( y + x \cdot \frac{dy}{dx} \right) + 3y^2 \cdot \frac{dy}{dx} &= 0 \\
-x \cdot \frac{dy}{dx} + 3y^2 \cdot \frac{dy}{dx} &= y - 3x^2 \\
3y^2 \cdot \frac{dy}{dx} - x \cdot \frac{dy}{dx} &= y - 3x^2 \\
(3y^2 - x) \frac{dy}{dx} &= y - 3x^2 \\
\frac{dy}{dx} &= \frac{y - 3x^2}{3y^2 - x}
\end{aligned}$$

4.  $x^2 \cos^2 y - \sin y = 0$

$$\begin{aligned}
\frac{d}{dx} (x^2 \cos^2 y - \sin y) &= \frac{d}{dx} (0) \\
2x \cos^2 y + x^2 \cdot \frac{d}{dx} (\cos^2 y) - \cos y \cdot \frac{dy}{dx} &= 0 \\
2x \cos^2 y + x^2 (2 \cos y) \cdot \frac{d}{dx} (\cos y) - \cos y \cdot \frac{dy}{dx} &= 0 \\
2x \cos^2 y + 2x^2 \cos y \cdot \frac{d}{dx} (\cos y) - \cos y \cdot \frac{dy}{dx} &= 0 \\
2x \cos^2 y + 2x^2 \cos y \cdot (-\sin y) \cdot \frac{dy}{dx} - \cos y \cdot \frac{dy}{dx} &= 0 \\
2x \cos^2 y - 2x^2 \sin y \cos y \cdot \frac{dy}{dx} - \cos y \cdot \frac{dy}{dx} &= 0 \\
-2x^2 \sin y \cos y \cdot \frac{dy}{dx} - \cos y \cdot \frac{dy}{dx} &= -2x \cos^2 y \\
2x^2 \sin y \cos y \cdot \frac{dy}{dx} + \cos y \cdot \frac{dy}{dx} &= 2x \cos^2 y \\
(2x^2 \sin y \cos y + \cos y) \frac{dy}{dx} &= 2x \cos^2 y \\
\frac{dy}{dx} &= \frac{2x \cos^2 y}{2x^2 \sin y \cos y + \cos y} \\
&= \frac{2x \cos y}{2x^2 \sin y + 1}
\end{aligned}$$

5.  $x^2 y^2 = 9$

$$\begin{aligned}
\frac{d}{dx} (x^2 y^2) &= \frac{d}{dx} (9) \\
2xy^2 + x^2 \cdot \frac{d}{dx} (y^2) &= 0 \\
2xy^2 + x^2 2y \cdot \frac{dy}{dx} &= 0
\end{aligned}$$

$$\begin{aligned}
2xy^2 + 2x^2y \cdot \frac{dy}{dx} &= 0 \\
xy^2 + x^2y \cdot \frac{dy}{dx} &= 0 \\
x^2y \cdot \frac{dy}{dx} &= -xy^2 \\
\frac{dy}{dx} &= -\frac{xy^2}{x^2y} \\
&= -\frac{y}{x}
\end{aligned}$$

6.  $2xy + \pi \sin y = 2\pi$

$$\begin{aligned}
\frac{d}{dx}(2xy + \pi \sin y) &= \frac{d}{dx}(2\pi) \\
2y + 2x \cdot \frac{dy}{dx} + \pi \cos y \cdot \frac{dy}{dx} &= 0 \\
2x \cdot \frac{dy}{dx} + \pi \cos y \cdot \frac{dy}{dx} &= -2y \\
(2x + \pi \cos y) \cdot \frac{dy}{dx} &= -2y \\
\frac{dy}{dx} &= -\frac{2y}{2x + \pi \cos y}
\end{aligned}$$

7.  $x^2y + xy^2 = 6$

$$\begin{aligned}
\frac{d}{dx}(x^2y + xy^2) &= \frac{d}{dx}(6) \\
2xy + x^2 \frac{dy}{dx} + y^2 + x \cdot \frac{d}{dx}(y^2) &= 0 \\
2xy + x^2 \frac{dy}{dx} + y^2 + x(2y) \frac{dy}{dx} &= 0 \\
x^2 \frac{dy}{dx} + 2xy \frac{dy}{dx} &= -2xy - y^2 \\
(x^2 + 2xy) \frac{dy}{dx} &= -2xy - y^2 \\
(x^2 + 2xy) \frac{dy}{dx} &= -(2xy + y^2) \\
\frac{dy}{dx} &= -\frac{2xy + y^2}{x^2 + 2xy}
\end{aligned}$$

8.  $x^2 = \frac{x-y}{x+y}$

$$\frac{d}{dx}(x^2) = \frac{d}{dx}\left(\frac{x-y}{x+y}\right)$$

$$\begin{aligned}
2x &= \frac{\left(1 - \frac{dy}{dx}\right)(x+y) - (x-y)\left(1 + \frac{dy}{dx}\right)}{(x+y)^2} \\
2x(x+y)^2 &= \left(1 - \frac{dy}{dx}\right)(x+y) - (x-y)\left(1 + \frac{dy}{dx}\right) \\
2x(x+y)^2 &= (x+y) - (x+y)\frac{dy}{dx} - (x-y) - (x-y)\frac{dy}{dx} \\
2x(x+y)^2 &= (x+y) - (x-y) - (x+y)\frac{dy}{dx} - (x-y)\frac{dy}{dx} \\
2x(x+y)^2 &= 2y - (x+y)\frac{dy}{dx} - (x-y)\frac{dy}{dx} \\
2x(x+y)^2 &= 2y - [(x+y) + (x-y)]\frac{dy}{dx} \\
2x(x+y)^2 &= 2y - 2x\frac{dy}{dx} \\
2x\frac{dy}{dx} &= 2y - 2x(x+y)^2 \\
x\frac{dy}{dx} &= y - x(x+y)^2 \\
\frac{dy}{dx} &= \frac{y}{x} - (x+y)^2
\end{aligned}$$

9.  $x \sin 2y = y \cos 2x$

$$\begin{aligned}
\frac{d}{dx}(x \sin 2y) &= \frac{d}{dx}(y \cos 2x) \\
\sin 2y + x \cdot \frac{d}{dx}(\sin 2y) &= \frac{dy}{dx} \cdot \cos 2x + y(-\sin 2x) \cdot \frac{d}{dx}(2x) \\
\sin 2y + x \cos 2y \cdot \frac{d}{dx}(2y) &= \frac{dy}{dx} \cdot \cos 2x - y \sin 2x \cdot (2) \\
\sin 2y + x \cos 2y \cdot 2\frac{dy}{dx} &= \frac{dy}{dx} \cdot \cos 2x - 2y \sin 2x \\
\sin 2y + 2x \cos 2y \cdot \frac{dy}{dx} &= \frac{dy}{dx} \cdot \cos 2x - 2y \sin 2x \\
2x \cos 2y \cdot \frac{dy}{dx} - \frac{dy}{dx} \cdot \cos 2x &= -\sin 2y - 2y \sin 2x \\
2x \cos 2y \cdot \frac{dy}{dx} - \cos 2x \cdot \frac{dy}{dx} &= -\sin 2y - 2y \sin 2x \\
-2x \cos 2y \cdot \frac{dy}{dx} + \cos 2x \cdot \frac{dy}{dx} &= \sin 2y + 2y \sin 2x \\
\cos 2x \cdot \frac{dy}{dx} - 2x \cos 2y \cdot \frac{dy}{dx} &= \sin 2y + 2y \sin 2x \\
(\cos 2x - 2x \cos 2y) \frac{dy}{dx} &= \sin 2y + 2y \sin 2x
\end{aligned}$$

$$\frac{dy}{dx} = \frac{\sin 2y + 2y \sin 2x}{\cos 2x - 2x \cos 2y}$$

10.  $x^2(x - y)^2 = x^2 - y^2$

$$\begin{aligned} \frac{d}{dx} (x^2(x - y)^2) &= \frac{d}{dx} (x^2 - y^2) \\ 2x(x - y)^2 + x^2 \cdot \frac{d}{dx}(x - y)^2 &= 2x - 2y \frac{dy}{dx} \\ 2x(x - y)^2 + x^2 \cdot 2(x - y) \cdot \frac{d}{dx}(x - y) &= 2x - 2y \frac{dy}{dx} \\ 2x(x - y)^2 + 2x^2(x - y) \cdot \frac{d}{dx}(x - y) &= 2x - 2y \frac{dy}{dx} \\ x(x - y)^2 + x^2(x - y) \cdot \frac{d}{dx}(x - y) &= x - y \frac{dy}{dx} \\ x(x - y)^2 + x^2(x - y) \left(1 - \frac{dy}{dx}\right) &= x - y \frac{dy}{dx} \\ x(x - y)^2 + x^2(x - y) - x^2(x - y) \frac{dy}{dx} &= x - y \frac{dy}{dx} \\ -x^2(x - y) \frac{dy}{dx} + y \frac{dy}{dx} &= x - x(x - y)^2 - x^2(x - y) \\ x^2(x - y) \frac{dy}{dx} - y \frac{dy}{dx} &= -x + x(x - y)^2 + x^2(x - y) \\ (x^2(x - y) - y) \frac{dy}{dx} &= x(x - y)^2 + x^2(x - y) - x \\ \frac{dy}{dx} &= \frac{x(x - y)^2 + x^2(x - y) - x}{x^2(x - y) - y} \end{aligned}$$

11.  $y^2 - 2x - 4y = 1$

$$\begin{aligned} \frac{d}{dx} (y^2 - 2x - 4y) &= \frac{d}{dx} (1) \\ 2y \frac{dy}{dx} - 2 - 4 \frac{dy}{dx} &= 0 \\ 2y \frac{dy}{dx} - 4 \frac{dy}{dx} &= 2 \\ y \frac{dy}{dx} - 2 \frac{dy}{dx} &= 1 \\ (y - 2) \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{y - 2} \end{aligned}$$