Millersville University
Department of Mathematics
MATH 161, *Calculus I*

Please use l'Hôpital's rule where appropriate to evaluate the following limits.

1. $\displaystyle\lim_{x\to\pi/2} \frac{1-\sin x}{1+\cos 2x}$

   Since

$$\begin{aligned}\lim_{x\to\pi/2}(1-\sin x) &= 0 \quad \text{and} \\ \lim_{x\to\pi/2}(1+\cos 2x) &= 0\end{aligned}$$

   the original limit is indeterminate of the form 0/0. According to l'Hôpital's rule

$$\begin{aligned}\lim_{x\to\pi/2} \frac{1-\sin x}{1+\cos 2x} &= \lim_{x\to\pi/2} \frac{-\cos x}{-2\sin 2x} \quad \text{(indeterminate } 0/0\text{)} \\ &= \lim_{x\to\pi/2} \frac{\sin x}{-4\cos 2x} \\ &= 0.\end{aligned}$$

2. $\displaystyle\lim_{x\to 1} \frac{x-1}{\ln x - \sin \pi x}$

   Since

$$\begin{aligned}\lim_{x\to 1}(x-1) &= 0 \quad \text{and} \\ \lim_{x\to 1}(\ln x - \sin \pi x) &= 0\end{aligned}$$

   the original limit is indeterminate of the form 0/0. According to l'Hôpital's rule

$$\begin{aligned}\lim_{x\to 1} \frac{x-1}{\ln x - \sin \pi x} &= \lim_{x\to 1} \frac{1}{\frac{1}{x} - \pi\cos \pi x} \\ &= \frac{1}{1+\pi}.\end{aligned}$$

3. $\displaystyle\lim_{x\to\infty} x \tan \frac{1}{x}$

   Since

$$\begin{aligned}\lim_{x\to\infty} x &= \infty \quad \text{and} \\ \lim_{x\to\infty} \tan \frac{1}{x} &= 0\end{aligned}$$

the original limit is indeterminate of the form $0 \cdot \infty$. We can rewrite the original limit in the form

$$\lim_{x\to\infty} \frac{\tan\frac{1}{x}}{\frac{1}{x}}$$

which is indeterminate of the form $0/0$. According to l'Hôpital's rule

$$\begin{aligned}
\lim_{x\to\infty} \frac{\tan\frac{1}{x}}{\frac{1}{x}} &= \lim_{x\to\infty} \frac{-\frac{1}{x^2}\sec^2\frac{1}{x}}{-\frac{1}{x^2}} \\
&= \lim_{x\to\infty} \sec^2\frac{1}{x} \\
&= 1
\end{aligned}$$

4. $\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{\sqrt{x}}\right)$

Since

$$\begin{aligned}
\lim_{x\to 0^+} \frac{1}{x} &= \infty \quad \text{and} \\
\lim_{x\to 0^+} \frac{1}{\sqrt{x}} &= \infty
\end{aligned}$$

the original limit is indeterminate of the form $\infty - \infty$. We can rewrite the original limit in the form

$$\lim_{x\to 0^+} \frac{1-\sqrt{x}}{x}$$

which is not an indeterminate form. Thus we cannot use l'Hôpital's rule.

$$\lim_{x\to 0^+} \frac{1-\sqrt{x}}{x} = \infty$$

5. $\lim_{x\to 1^+} (x^2 - 2x + 1)^{x-1}$

Since

$$\begin{aligned}
\lim_{x\to 1^+} (x^2 - 2x + 1) &= 0 \quad \text{and} \\
\lim_{x\to 1^+} (x - 1) &= 0
\end{aligned}$$

the original limit is indeterminate of the form 0^0. We will let

$$\begin{aligned}
y &= (x^2 - 2x + 1)^{x-1} \\
\ln y &= \ln(x^2 - 2x + 1)^{x-1} \\
&= (x-1)\ln(x^2 - 2x + 1) \\
&= \frac{\ln(x^2 - 2x + 1)}{\frac{1}{x-1}}
\end{aligned}$$

As $x \to 1^+$ the last expression is indeterminate of the form ∞/∞. According to l'Hôpital's rule

$$\begin{aligned}
\lim_{x\to 1^+} \frac{\ln(x^2-2x+1)}{\frac{1}{x-1}} &= \lim_{x\to 1^+} \frac{\frac{2x-2}{x^2-2x+1}}{\frac{-1}{(x-1)^2}} \\
&= \lim_{x\to 1^+} 2-2x \\
&= 0.
\end{aligned}$$

Thus

$$\lim_{x\to 1^+} (x^2-2x+1)^{x-1} = e^0 = 1$$

6. $\lim_{x\to 1} x^{\frac{1}{1-x}}$

Since

$$\lim_{x\to 1} x = 1 \quad \text{and} \quad \lim_{x\to 1} \frac{1}{1-x} \quad \text{does not exist}$$

we will have to resort to evaluating the one-sided limits and seeing if they are equal.

$$\begin{aligned}
\lim_{x\to 1^+} x &= 1 \quad \text{and} \\
\lim_{x\to 1^+} \frac{1}{1-x} &= \infty
\end{aligned}$$

thus

$$\lim_{x\to 1^+} x^{\frac{1}{1-x}}$$

is indeterminate of the form 1^∞. We will let

$$\begin{aligned}
y &= x^{\frac{1}{1-x}} \\
\ln y &= x^{\frac{1}{1-x}} \\
&= \frac{1}{1-x} \ln x \\
&= \frac{\ln x}{1-x}
\end{aligned}$$

which is indeterminate of the form $0/0$. According to l'Hôpital's rule

$$\begin{aligned}
\lim_{x\to 1^+} \frac{\ln x}{1-x} &= \lim_{x\to 1^+} \frac{\frac{1}{x}}{-1} \\
&= \lim_{x\to 1^+} -\frac{1}{x} \\
&= -1
\end{aligned}$$

Thus

$$\lim_{x\to 1^+} x^{\frac{1}{1-x}} = e^{-1}$$

Now we must consider the limit from the left.

$$\begin{aligned}\lim_{x\to 1^-} x &= 1 \quad \text{and} \\ \lim_{x\to 1^-} \frac{1}{1-x} &= -\infty\end{aligned}$$

thus

$$\lim_{x\to 1^-} x^{\frac{1}{1-x}}$$

is indeterminate of the form 1^∞. We will let

$$\begin{aligned}y &= x^{\frac{1}{1-x}} \\ \ln y &= x^{\frac{1}{1-x}} \\ &= \frac{1}{1-x}\ln x \\ &= \frac{\ln x}{1-x}\end{aligned}$$

which is indeterminate of the form $0/0$. According to l'Hôpital's rule

$$\begin{aligned}\lim_{x\to 1^-} \frac{\ln x}{1-x} &= \lim_{x\to 1^-} \frac{\frac{1}{x}}{-1} \\ &= \lim_{x\to 1^-} -\frac{1}{x} \\ &= -1\end{aligned}$$

Thus

$$\lim_{x\to 1^-} x^{\frac{1}{1-x}} = e^{-1}$$

Since the one-sided limits exist and agree then the original limit exists and

$$\lim_{x\to 1} x^{\frac{1}{1-x}} = e^{-1}$$

7. $\lim_{x\to\infty} (1+2x)^{\frac{1}{2\ln x}}$

Since

$$\begin{aligned}\lim_{x\to\infty} (1+2x) &= \infty \quad \text{and} \\ \lim_{x\to\infty} \frac{1}{2\ln x} &= 0\end{aligned}$$

the original limit is indeterminate of the form ∞^0. We will let

$$\begin{aligned}y &= (1+2x)^{\frac{1}{2\ln x}} \\ \ln y &= \ln(1+2x)^{\frac{1}{2\ln x}} \\ &= \frac{1}{2\ln x}\ln(1+2x) \\ &= \frac{\ln(1+2x)}{2\ln x}\end{aligned}$$

As $x \to \infty$ the last expression is indeterminate of the form ∞/∞. According to l'Hôpital's rule

$$\begin{aligned}\lim_{x\to\infty} \frac{\ln(1+2x)}{2\ln x} &= \lim_{x\to\infty} \frac{\frac{2}{1+2x}}{\frac{2}{x}} \\ &= \lim_{x\to\infty} \frac{x}{1+2x} \quad \text{(indeterminate } \infty/\infty) \\ &= \lim_{x\to\infty} \frac{1}{2} \\ &= \frac{1}{2}\end{aligned}$$

Thus

$$\lim_{x\to\infty} (1+2x)^{\frac{1}{2\ln x}} = e^{1/2} = \sqrt{e}.$$

8. $\lim_{x\to\infty} x^2 e^{-x}$

Since we mat rewrite the expression x^2e^{-x} as $\frac{x^2}{e^x}$ we will consider the limit

$$\lim_{x\to\infty} \frac{x^2}{e^x},$$

which is indeterminate of the form ∞/∞ because

$$\begin{aligned}\lim_{x\to\infty} x^2 &= \infty \quad \text{and} \\ \lim_{x\to\infty} e^x &= \infty.\end{aligned}$$

According to l'Hôpital's rule

$$\begin{aligned}\lim_{x\to\infty} \frac{x^2}{e^x} &= \lim_{x\to\infty} \frac{2x}{e^x} \quad \text{(indeterminate } \infty/\infty) \\ &= \lim_{x\to\infty} \frac{2}{e^x} \\ &= 0\end{aligned}$$

9. $\lim_{x\to\infty} \int_x^{2x} \frac{1}{t}\, dt$

Evaluating the definite integral first produces

$$\int_x^{2x} \frac{1}{t}\, dt = \ln t|_x^{2x} = \ln 2x - \ln x = \ln 2 + \ln x - \ln x = \ln 2.$$

Thus

$$\lim_{x\to\infty} \int_x^{2x} \frac{1}{t}\, dt = \lim_{x\to\infty} \ln 2 = \ln 2.$$

10. $\displaystyle\lim_{x\to\infty} \frac{1}{x\ln x}\int_1^x \ln t\,dt$

Evaluating the definite integral first produces

$$\int_1^x \ln t\,dt = \left.(t\ln t - t)\right|_1^x = x\ln x - x + 1.$$

Thus

$$\lim_{x\to\infty} \frac{1}{x\ln x}\int_1^x \ln t\,dt = \lim_{x\to\infty} \frac{1}{x\ln x}\cdot(x\ln x - x + 1) = \lim_{x\to\infty} \frac{x\ln x - x + 1}{x\ln x} = \lim_{x\to\infty}\left(1 - \frac{x-1}{x\ln x}\right).$$

Since

$$\begin{aligned} \lim_{x\to\infty}(x-1) &= \infty \quad \text{and} \\ \lim_{x\to\infty} x\ln x &= \infty, \end{aligned}$$

The limit is indeterminate of the form ∞/∞. According to l'Hôpital's rule

$$\begin{aligned} \lim_{x\to\infty}\left(1 - \frac{x-1}{x\ln x}\right) &= 1 - \lim_{x\to\infty}\frac{x-1}{x\ln x} \\ &= 1 - \lim_{x\to\infty}\frac{1}{1+\ln x} \\ &= 1. \end{aligned}$$