Millersville University
Department of Mathematics
MATH 161, *Calculus I* **Final Exam Review**
April 22, 2004

The following exercises will help you prepare for the final examination. Solutions will be available on Monday, April 26, 2004.

1. Evaluate the following limits if they exist. If a limit does not exist, please explain why.

   (a) $\lim_{x\to\infty} \frac{\sqrt{2x^2-x}}{6x-1}$
   Since this is a limit involving $x \to \infty$, multiply the numerator and denominator by the reciprocal of the highest power of x in the denominator.

$$\begin{aligned}
\lim_{x\to\infty} \frac{\sqrt{2x^2-x}}{6x-1} &= \lim_{x\to\infty} \frac{\sqrt{2x^2-x}}{6x-1} \cdot \frac{1/x}{1/x} \\
&= \lim_{x\to\infty} \frac{\sqrt{2x^2-x}}{6x-1} \cdot \frac{\sqrt{1/x^2}}{1/x} \\
&= \lim_{x\to\infty} \frac{\sqrt{(2x^2-x)\frac{1}{x^2}}}{(6x-1)\frac{1}{x}} \\
&= \lim_{x\to\infty} \frac{\sqrt{2-\frac{1}{x}}}{6-\frac{1}{x}} \\
&= \frac{\sqrt{2}}{6}
\end{aligned}$$

   (b) $\lim_{x\to 3^-} \frac{\sqrt{x}-\sqrt{3}}{x-3}$
   Multiply numerator and denominator by the conjugate of the numerator.

$$\begin{aligned}
\lim_{x\to 3^-} \frac{\sqrt{x}-\sqrt{3}}{x-3} &= \lim_{x\to 3^-} \left(\frac{\sqrt{x}-\sqrt{3}}{x-3}\right)\left(\frac{\sqrt{x}+\sqrt{3}}{\sqrt{x}+\sqrt{3}}\right) \\
&= \lim_{x\to 3^-} \frac{(\sqrt{x}-\sqrt{3})(\sqrt{x}+\sqrt{3})}{(x-3)(\sqrt{x}+\sqrt{3})} \\
&= \lim_{x\to 3^-} \frac{x-3}{(x-3)(\sqrt{x}+\sqrt{3})} \\
&= \lim_{x\to 3^-} \frac{1}{\sqrt{x}+\sqrt{3}} \\
&= \frac{1}{\sqrt{3}+\sqrt{3}} = \frac{1}{2\sqrt{3}}
\end{aligned}$$

2. Find the value of c which makes the following function continuous at $x = -2$:

$$f(x) = \begin{cases} \dfrac{x+10}{4} & \text{if } x \le -2 \\ \dfrac{c^3}{18} + x + 1 & \text{if } x > -2 \end{cases}$$

   If $f(x)$ is continuous at $x = -2$ then by definition

$$\lim_{x\to -2^-} f(x) = \lim_{x\to -2^+} f(x),$$

thus we must have

$$\begin{aligned}
\lim_{x\to -2^-} f(x) &= \lim_{x\to -2^-} \frac{x+10}{4} = \frac{-2+10}{4} = 2 \\
\lim_{x\to -2^+} f(x) &= \lim_{x\to -2^+} \frac{c^3}{18} + x + 1 = \frac{c^3}{18} + (-2) + 1 = \frac{c^3}{18} - 1.
\end{aligned}$$

Solving the following equation,

$$2 = \frac{c^3}{18} - 1$$

yields $c = 3\sqrt[3]{2}$.

3. Find the derivatives of the following functions.

(a) $f(x) = e^{x^3} + \sin(e^x + x^2) + \ln(2 + \cos x)$

Here we will use the chain rule on each of the three expressions added to form $f(x)$.

$$\begin{aligned}
f'(x) &= e^{x^3}\left(\frac{d}{dx}x^3\right) + \cos(e^x + x^2)\left(\frac{d}{dx}(e^x + x^2)\right) + \frac{1}{2+\cos x}\left(\frac{d}{dx}(2+\cos x)\right) \\
&= 3x^2 e^{x^3} + (e^x + 2x)\cos(e^x + x^2) - \frac{\sin x}{2+\cos x}
\end{aligned}$$

(b) $g(x) = \left(1 + \frac{1}{x^2} + \frac{1}{x^2+1}\right)^2$

Here we will use the chain rule and properties of exponents to conveniently take the derivative of $g(x)$.

$$\begin{aligned}
g'(x) &= 2\left(1 + \frac{1}{x^2} + \frac{1}{x^2+1}\right)\left(\frac{d}{dx}\left[1 + \frac{1}{x^2} + \frac{1}{x^2+1}\right]\right) \\
&= 2\left(1 + \frac{1}{x^2} + \frac{1}{x^2+1}\right)\left(\frac{d}{dx}\left[1 + x^{-2} + (x^2+1)^{-1}\right]\right) \\
&= 2\left(1 + \frac{1}{x^2} + \frac{1}{x^2+1}\right)\left(-2x^{-3} - (x^2+1)^{-2}\left[\frac{d}{dx}(x^2+1)\right]\right) \\
&= 2\left(1 + \frac{1}{x^2} + \frac{1}{x^2+1}\right)\left(-\frac{2}{x^3} - \frac{2x}{(x^2+1)^2}\right) \\
&= -4\left(1 + \frac{1}{x^2} + \frac{1}{x^2+1}\right)\left(\frac{1}{x^3} + \frac{x}{(x^2+1)^2}\right)
\end{aligned}$$

(c) $h(x) = (x^2 - 1)^{15}(x^4 - 3x)$

Here we will use the chain rule and product rule for derivatives to differentiate $h(x)$.

$$\begin{aligned}
h'(x) &= 15(x^2-1)^{14}\left(\frac{d}{dx}(x^2-1)\right)(x^4-3x) + (x^2-1)^{15}(4x^3-3) \\
&= 15(x^2-1)^{14}(2x)(x^4-3x) + (x^2-1)^{15}(4x^3-3) \\
&= (x^2-1)^{14}(30x(x^4-3x) + (x^2-1)(4x^3-3)) \\
&= (x^2-1)^{14}(30x^5 - 90x^2 + 4x^5 - 4x^3 - 3x^2 + 3) \\
&= (x^2-1)^{14}(34x^5 - 4x^3 - 93x^2 + 3)
\end{aligned}$$

(d) $F(x) = \int_0^{\cos x} \sqrt{t^3 + t + 3}\, dt$

Here we will use the chain rule, thinking of $F(x) = f(g(x))$ where

$$f(x) = \int_0^x \sqrt{t^3 + t + 3}\, dt \qquad \text{and} \qquad g(x) = \cos x,$$

and the second part of the Fundamental Theorem of Calculus.

$$\begin{aligned} F'(x) &= \sqrt{\cos^3 x + \cos x + 3}\left(\frac{d}{dx}\cos x\right) \\ &= -(\sin x)\sqrt{\cos^3 x + \cos x + 3} \end{aligned}$$

(e) $G(x) = (x+1)^{\sin x}$

We will use logarithmic differentiation.

$$\begin{aligned} y &= (x+1)^{\sin x} \\ \ln y &= \ln\left((x+1)^{\sin x}\right) \\ &= (\sin x)\ln(x+1) \\ \frac{d}{dx}(\ln y) &= \frac{d}{dx}((\sin x)\ln(x+1)) \\ \frac{1}{y}\frac{dy}{dx} &= (\cos x)\ln(x+1) + (\sin x)\frac{1}{x+1} \\ \frac{dy}{dx} &= y\left((\cos x)\ln(x+1) + \frac{\sin x}{x+1}\right) \\ G'(x) &= (x+1)^{\sin x}\left((\cos x)\ln(x+1) + \frac{\sin x}{x+1}\right) \end{aligned}$$

4. Use the definition of the derivative as the limit of a difference quotient to find the derivative of $f(x) = \sqrt{x+1}$.

By definition,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

if the limit exists. Thus if $f(x) = \sqrt{x+1}$, then

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{\sqrt{x+h+1} - \sqrt{x+1}}{h} \\ &= \lim_{h\to 0}\frac{(\sqrt{x+h+1} - \sqrt{x+1})}{h}\cdot\frac{(\sqrt{x+h+1} + \sqrt{x+1})}{(\sqrt{x+h+1} + \sqrt{x+h})} \\ &= \lim_{h\to 0}\frac{(x+h+1) - (x+1)}{h(\sqrt{x+h+1} + \sqrt{x+1})} \\ &= \lim_{h\to 0}\frac{h}{h(\sqrt{x+h+1} + \sqrt{x+1})} \\ &= \lim_{h\to 0}\frac{1}{\sqrt{x+h+1} + \sqrt{x+1}} \\ &= \frac{1}{\sqrt{x+1} + \sqrt{x+1}} \\ f'(x) &= \frac{1}{2\sqrt{x+1}} \end{aligned}$$

5. Use linear approximation to approximate the value of $e^{1/10}$.

We will let $f(x) = e^x$. Since the real numbers 0 and 1/10 are close to each other on the number line and since we know $f(0) = 1$ we can use the linear approximation to $f(x)$ at $x_0 = 0$. The linear approximation is therefore

$$\begin{aligned} L(x) &= f'(x_0)(x - x_0) + f(x_0) \\ &= e^{x_0}(x - x_0) + e^{x_0} \\ &= e^0(x - 0) + e^0 \\ &= x + 1 \end{aligned}$$

Thus we have $L(1/10) = 1 + 1/10 = 11/10 \approx e^{1/10}$.

6. Find the equation of the tangent line to the curve

$$(2x - y)^3 + 6x = 2y + y^2 - 2$$

at the point $(2, 3)$.

We must use implicit differentiation to find the slope of the tangent line.

$$\begin{aligned}
\frac{d}{dx}\left((2x-y)^3 + 6x\right) &= \frac{d}{dx}\left(2y + y^2 - 2\right) \\
3(2x-y)^2\left(\frac{d}{dx}(2x-y)\right) + 6 &= 2\frac{dy}{dx} + 2y\frac{dy}{dx} \\
3(2x-y)^2(2 - \frac{dy}{dx}) + 6 &= 2(1+y)\frac{dy}{dx} \\
6(2x-y)^2 - 3(2x-y)^2\frac{dy}{dx} + 6 &= 2(1+y)\frac{dy}{dx} \\
6(2x-y)^2 + 6 &= \left(2(1+y) + 3(2x-y)^2\right)\frac{dy}{dx} \\
6((2x-y)^2 + 1) &= \left(2(1+y) + 3(2x-y)^2\right)\frac{dy}{dx} \\
\frac{dy}{dx} &= \frac{6((2x-y)^2+1)}{2(1+y) + 3(2x-y)^2}
\end{aligned}$$

Thus the slope of the tangent line is $m = 12/11$. The equation of the tangent line is then

$$\begin{aligned}
\frac{12}{11} &= \frac{y-3}{x-2} \\
y &= \frac{12}{11}x + \frac{9}{11}.
\end{aligned}$$

7. Evaluate the following definite and indefinite integrals.

(a) $\displaystyle\int_0^1 (x^2+2)\sqrt{x^3+6x+5}\,dx$

Use integration by substitution with

$$\begin{aligned}
u &= x^3 + 6x + 5 \\
du &= (3x^2 + 6)\,dx \\
\frac{1}{3}\,du &= (x^2+2)dx
\end{aligned}$$

Then

$$\begin{aligned}
\int_0^1 (x^2+2)\sqrt{x^3+6x+5}\,dx &= \int_5^{12} \frac{1}{3}\sqrt{u}\,du \\
&= \frac{1}{3}\int_5^{12} u^{1/2}\,du \\
&= \frac{1}{3}\cdot\frac{2}{3}u^{3/2}\Big|_5^{12} \\
&= \frac{2}{9}\left(12^{3/2} - 5^{3/2}\right) \\
&= \frac{2}{9}\left(12\sqrt{12} - 5\sqrt{5}\right)
\end{aligned}$$

(b) $\int \frac{x+1}{x+2}\,dx$

Use integration by substitution with

$$\begin{aligned} u &= x+2 \\ u-1 &= x+1 \\ du &= dx \end{aligned}$$

Then we have

$$\begin{aligned} \int \frac{x+1}{x+2}\,dx &= \int \frac{u-1}{u}\,du \\ &= \int \left(1-\frac{1}{u}\right)\,du \\ &= u - \ln|u| + C \\ &= (x+2) - \ln|x+2| + C. \end{aligned}$$

(c) $\int \frac{\sec^2 x}{2+\tan x}\,dx$

Use integration by substitution with

$$\begin{aligned} u &= 2+\tan x \\ du &= \sec^2 x\,dx \end{aligned}$$

Then we have

$$\begin{aligned} \int \frac{\sec^2 x}{2+\tan x}\,dx &= \int \frac{1}{u}\,du \\ &= \ln|u| + C \\ &= \ln|2+\tan x| + C \end{aligned}$$

(d) $\int_0^2 \frac{x^3}{\sqrt{x^4+9}}\,dx$

Use integration by substitution with

$$\begin{aligned} u &= x^4+9 \\ du &= 4x^3\,dx \\ \frac{1}{4}\,du &= x^3\,dx \end{aligned}$$

Then we have

$$\begin{aligned} \int_0^2 \frac{x^3}{\sqrt{x^4+9}}\,dx &= \int_9^{25} \frac{1}{4}\cdot\frac{1}{\sqrt{u}}\,du \\ &= \frac{1}{4}\int_9^{25} u^{-1/2}\,du \\ &= \frac{1}{4}\cdot 2u^{1/2}\Big|_9^{25} \\ &= \frac{1}{2}\left(25^{1/2} - 9^{1/2}\right) \\ &= \frac{1}{2}(5-3) \\ &= 1 \end{aligned}$$

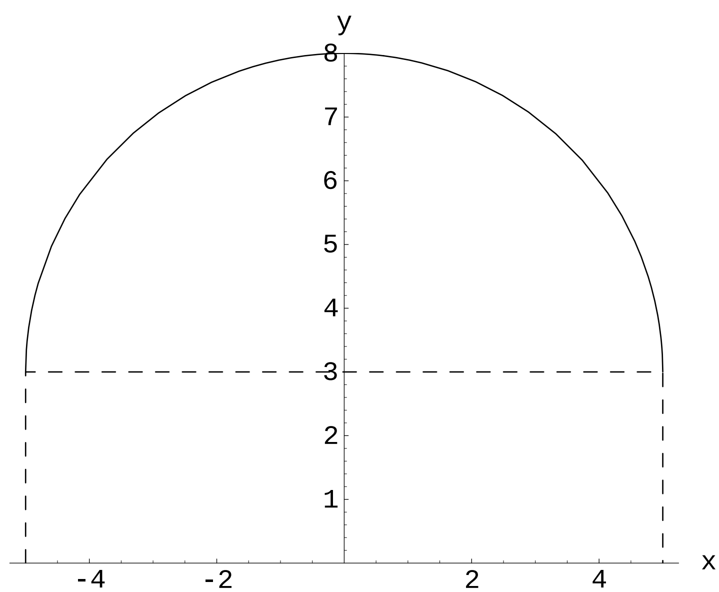
(e) $\int_1^4 \frac{x^2 - x + 1}{\sqrt{x}}\, dx$

$$
\begin{aligned}
\int_1^4 \frac{x^2 - x + 1}{\sqrt{x}}\, dx &= \int_1^4 \frac{x^2}{\sqrt{x}} - \frac{x}{\sqrt{x}} + \frac{1}{\sqrt{x}}\, dx \\
&= \int_1^4 x^{3/2} - x^{1/2} + x^{-1/2}\, dx \\
&= \frac{2}{5}x^{5/2} - \frac{2}{3}x^{3/2} + 2x^{1/2}\Big|_1^4 \\
&= \frac{2}{5}(4^{5/2} - 1^{5/2}) - \frac{2}{3}(4^{3/2} - 1^{3/2}) + 2(4^{1/2} - 1^{1/2}) \\
&= \frac{2}{5}(32 - 1) - \frac{2}{3}(8 - 1) + 2(2 - 1) \\
&= \frac{62}{5} - \frac{14}{3} + 2 \\
&= \frac{146}{15}
\end{aligned}
$$

(f) $\int_{-5}^{5} 3 + \sqrt{25 - x^2}\, dx$

If we graph the integrand $y = 3 + \sqrt{25 - x^2}$, we see a rectangle with base 10 and height 3 topped by a half circle of radius 5. The definite integral is the area of this region in the xy-plane. Thus

$$\int_{-5}^{5} 3 + \sqrt{25 - x^2}\, dx = (3)(10) + \frac{1}{2}\pi(5)^2 = 30 + \frac{25\pi}{2}$$



8. Suppose that f is a continuous function such that $f(3) = -2$ and $f(7) = 3$. Show that the equation $\frac{1}{1+x} f(x) = 0$ for some x between 3 and 7.

   Since f is continuous and $\frac{1}{x+1}$ is continuous on the interval $[3, 7]$ then the function $g(x) = \frac{1}{x+1} f(x)$ is continuous on the interval $[3, 7]$. Note that $g(3) = -1/2 < 0 < 3/8 = g(7)$. Thus by the Intermediate Value Theorem the equation $g(x) = 0$ has a solution for some $3 < x < 7$.

9. A bagel is placed in a 50° cooler. After 6 minutes, the bagel's temperature is 70°; after 9 minutes, the bagel's temperature is 60°. What was the bagel's initial temperature?

We will assume the bagel and the cooler are obeying Newton's Law of Cooling,

$$\frac{dT}{dt} = r(T - K)$$

where T is the temperature of the bagel, t is time measured in minutes, and K is the constant temperature of the cooler. The parameter r is a constant. We would like to solve this differential equation for the function $T(t)$.

$$\begin{aligned}
\frac{dT}{dt} &= r(T-50) \\
\frac{1}{T-50}dT &= r\,dt \\
\int \frac{1}{T-50}dT &= \int r\,dt \\
\ln|T-50| &= rt + C \\
e^{\ln(T-50)} &= e^{rt+C} \\
T - 50 &= Ce^{rt} \\
T(t) &= 50 + Ce^{rt}
\end{aligned}$$

We know $70 = 50 + Ce^{6r}$ which implies $20 = Ce^{6r}$. We also know $60 = 50 + Ce^{9r}$ which implies $10 = Ce^{9r}$. If we divide the equation $20 = Ce^{6r}$ by the equation $10 = Ce^{9r}$ the constant C cancels out and we have

$$\begin{aligned}
2 &= e^{-3r} \\
\ln 2 &= -3r \\
r &= -\frac{1}{3}\ln 2
\end{aligned}$$

Substituting this value of r into the equation $70 = 50 + Ce^{6r}$ we get

$$\begin{aligned}
70 &= 50 + Ce^{6\left(-\frac{1}{3}\ln 2\right)} \\
20 &= Ce^{-2\ln 2} \\
C &= 20e^{2\ln 2} \\
&= 20e^{\ln 4} \\
C &= 80
\end{aligned}$$

Thus at time $t = 0$, when the bagel was placed in the cooler,

$$T(0) = 50 + 80e^{r(0)} = 50 + 80 = 130°\text{F}.$$

10. A rectangular box with a square bottom and no top is to have a volume of 216 cubic feet. The material for the bottom costs \$8 per square foot, while the material for the sides costs \$4 per square foot. Find the dimensions which give the box with smallest total cost.

    Since the bottom of the box is a square, we will let its dimensions be x feet by x feet. The height of the box will be y feet. By assumption the box is to have a volume of 216 feet, thus

$$x^2y = 216 \quad \text{which implies} \quad y = \frac{216}{x^2}.$$

If the material for the bottom of the box costs \$8 per square foot, then the cost of the bottom of the box is $8x^2$ dollars. Since the four sides of the box are rectangles with base dimension x and height dimension y and the cost of the material for the sides is \$4 per square foot, then the cost of a single side is $4xy$ dollars. Thus the function which describes the total cost of the materials for the box is

$$\begin{aligned} C &= 8x^2 + 4(4xy) \\ &= 8x^2 + 16x\frac{216}{x^2} \\ &= 8x^2 + \frac{3456}{x} \end{aligned}$$

We can minimize C by taking its derivative.

$$\frac{dC}{dx} = 16x - \frac{3456}{x^2}$$

The critical numbers occur where this derivative is zero or undefined. dC/dx is undefined when $x = 0$, but this problem requires that all the dimensions of the box be positive, so we can ignore this critical number. Setting the derivative equal to zero and solving for x,

$$\begin{aligned} 0 &= 16x - \frac{3456}{x^2} \\ 16x &= \frac{3456}{x^2} \\ x &= \frac{216}{x^2} \\ x^3 &= 216 \\ x &= 6 \end{aligned}$$

According to either the first derivative test or the second derivative test, the cost function has a minimum at $x = 6$. Thus the dimensions of the box with the smallest cost are $x = 6$ and $y = 6$.

11. Graph the function $f(x) = \frac{e^x}{x-1}$. Specifically:

- Find the x-coordinates of the x-intercepts and the y-coordinate of the y-intercept (if any).

  This function has no x-intercepts since $e^x \neq 0$ for all x. The y-intercept of this function is found at the coordinates $(0, f(0)) = (0, -1)$.

- Find the intervals on which f increases and the intervals on which f decreases.

  Information about increasing and decreasing is found in the first derivative, which for this function is

  $$f'(x) = \frac{e^x(x-1) - e^x(1)}{(x-1)^2} = \frac{e^x((x-1)-1)}{(x-1)^2} = \frac{e^x(x-2)}{(x-1)^2}.$$

  Since this derivative can only change sign where the expression $(x - 2)$ changes sign (since e^x is always positive, and $(x-1)^2$ is positive except at $x = 1$), then we can say:

  $f(x)$ is increasing on the interval $(2, \infty)$, and

  $f(x)$ is decreasing on the interval $(-\infty, 1) \bigcup (1, 2)$.

- Find the x-coordinates of any local maxima or minima.

  The critical numbers of the function occur where the derivative is zero or undefined. This derivative is zero only at $x = 2$. We can ignore the value $x = 1$ since this value is not in the domain of the original function $f(x)$. According to the first derivative test, $f(x)$ has a local minimum at $x = 2$. The coordinates of this local minimum are $(2, f(2)) = (2, e^2)$.

- Find the intervals on which f is concave up and the intervals on which f is concave down.

  Information about the concavity of a function found in the second derivative, which for this function is

  $$f''(x) = \frac{(e^x(x-2) + e^x)(x-1)^2 - e^x(x-2)2(x-1)}{(x-1)^4}$$

$$
\begin{aligned}
&= \frac{(e^x(x-2)+e^x)(x-1)-2e^x(x-2)}{(x-1)^3} \\
&= \frac{e^x(x-2+1)(x-1)-2e^x(x-2)}{(x-1)^3} \\
&= \frac{e^x(x-1)^2-2e^x(x-2)}{(x-1)^3} \\
&= \frac{e^x((x-1)^2-2(x-2))}{(x-1)^3} \\
&= \frac{e^x(x^2-4x+5)}{(x-1)^3}
\end{aligned}
$$

The quadratic expression $x^2 - 4x + 5$ is always positive as we can check using the quadratic formula. Thus the second derivative can only change sign where the expression $(x - 1)$ changes sign (since e^x is always positive). Thus we can say:

$f(x)$ is concave up on the interval $(1, \infty)$, and

$f(x)$ is concave down on the interval $(-\infty, 1)$.

- Find the x-coordinates of any inflection points.

  An inflection point occurs at a place where the function is continuous and the concavity changes. The concavity changes at $x = 1$, but the function is undefined there, thus $f(x)$ has no points of inflection.

- Locate any vertical asymptotes, and compute the relevant limits.

  The function has a vertical asymptote at $x = 1$, since

$$
\begin{aligned}
\lim_{x\to 1^-} f(x) &= -\infty, \\
\lim_{x\to 1^+} f(x) &= \infty.
\end{aligned}
$$

- Locate any horizontal asymptotes, and compute the relevant limits.

  The function has a horizontal asymptotes at $y = 0$.

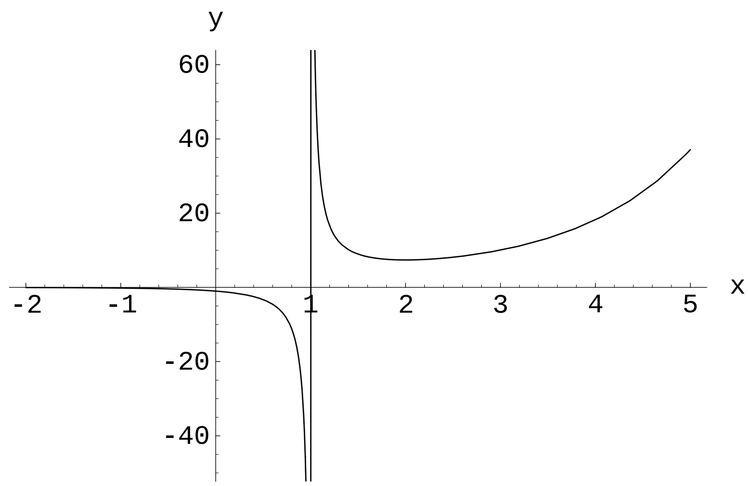
$$
\lim_{x\to-\infty} \frac{e^x}{x-1} = \frac{\lim_{x\to-\infty} e^x}{\lim_{x\to-\infty}(x-1)} = \frac{0}{-\infty} = 0
$$

  The limit as x approaches ∞ is difficult to determine, since

$$
\lim_{x\to\infty} \frac{e^x}{x-1} = \lim_{x\to\infty} \frac{e^x}{(x-1)} \cdot \frac{1/x}{1/x} = \lim_{x\to\infty} \frac{e^x/x}{1-1/x} = \lim_{x\to\infty} \frac{e^x}{x}
$$

  We will learn how to evaluate such limits in the second semester of calculus. However we can also realize from what we have already determined that on the interval $(2, \infty)$ the function is increasing and also concave up, in other words the function is increasing at an increasing rate and therefore could not “flatten out” and form a horizontal asymptote.

- Make a qualitatively accurate sketch of the graph based on the information above.



y
60
40
20
-20
-40
-2
-1
1
2
3
4
5
x