1. (15 points) The US Postal Service regulations governing the size of boxes that can be shipped state that the length of the box plus the girth (the distance around) of the box cannot be greater than 108 inches. Find the dimensions of the largest box with a square end that can be shipped.

Let the box have dimensions $x \times x \times l$, where $x$ is the length of edge of the square end of the box and $l$ is the length of the box. The girth of the box is then $4x$. The constraint equation for this problem is then that $4x + l = 108$. The volume of the box
is $V = x^2 l = x^2 (108 - 4x)$. The maximum volume of the box will occur at a critical value for $x$. Due to the constraint equation we know that $0 \leq x \leq \frac{108}{4} = 27$.

\[
V'(x) = 216x - 12x^2 \quad \text{and thus}
\]
\[
0 = 216x - 12x^2
\]
\[
0 = 12x(18 - x)
\]

Therefore the critical numbers are $x = 0$ and $x = 18$. Due to physical considerations the length of the edge of the box cannot equal zero, thus the only critical value of $x$ we must consider is $x = 18$. Since $V''(x) = 216 - 24x$ and $V''(18) < 0$ then according to the second derivative test, the volume is maximized when $x = 18$. Thus the dimensions of the largest box which can be mailed are $18 \times 18 \times 36$. 

2. (10 points each) Evaluate the following definite and indefinite integrals.

(a) \( \int x^{1/3} \sec^2(2 - x^{4/3}) \, dx \)

Let \( u = 2 - x^{4/3} \), then \( du = -\frac{4}{3}x^{1/3} \, dx \) or equivalently \( -\frac{3}{4} du = x^{1/3} \, dx \). Substituting these expressions into the indefinite integral above we get

\[
\int x^{1/3} \sec^2(2 - x^{4/3}) \, dx = \int -\frac{3}{4} \sec^2 u \, du = -\frac{3}{4} \tan u + C = -\frac{3}{4} \tan(2 - x^{4/3}) + C
\]

(b) \( \int \frac{2x^2}{\sqrt{15x^3 + 4}} \, dx \)

Let \( u = 15x^3 + 4 \), then \( du = 45x^2 \, dx \) or equivalently \( \frac{2}{45} du = 2x^2 \, dx \). Substituting these expressions into the indefinite integral above we get

\[
\int \frac{2x^2}{\sqrt{15x^3 + 4}} \, dx = \int \frac{2}{45} \frac{1}{\sqrt{u}} \, du = \frac{2}{45} \int u^{-1/2} \, du = \frac{2}{45} 2u^{1/2} + C = \frac{4}{45} u^{1/2} + C = \frac{4}{45} (15x^3 + 4)^{1/2} + C = \frac{4}{45} \sqrt{15x^3 + 4} + C
\]

(c) \( \int_0^2 \frac{1}{3x + 4} \, dx \)

Let \( u = 3x + 4 \), then \( du = 3 \, dx \) or equivalently \( \frac{1}{3} du = dx \). Substituting these expressions into the definite integral above we get

\[
\int_0^2 \frac{1}{3x + 4} \, dx = \int_{3(0)+4}^{3(2)+4} \frac{1}{3} \frac{1}{u} \, du = \int_4^{10} \frac{1}{3} \frac{1}{u} \, du = \frac{1}{3} \int_4^{10} \frac{1}{u} \, du
\]
\[= \frac{1}{3} \ln |u|_{10}^{4} \]
\[= \frac{1}{3} \ln 10 - \ln 4 \]
\[= \frac{1}{3} \ln \frac{10}{4} \]
\[= \frac{1}{3} \ln \frac{5}{2} \]

(d) \[\int_{\pi/2}^{\pi} \cos^3 x \sin x \, dx \]

Let \( u = \cos x \), then \( du = -\sin x \, dx \) or equivalently \( -du = \sin x \, dx \). Substituting these expressions into the definite integral above we get

\[\int_{\pi/2}^{\pi} \cos^3 x \sin x \, dx = \int_{\cos \pi/2}^{\cos \pi} u^3 \, du \]
\[= \int_{0}^{1} u^3 \, du \]
\[= \frac{1}{4} u^4 \bigg|_{0}^{1} \]
\[= \frac{1}{4} ((0)^4 - (-1)^4) \]
\[= \frac{1}{4} (0 - 1) \]
\[= -\frac{1}{4} \]

(e) \[\int_{\ln 2}^{1} e^x \cos(e^x - 2) \, dx \]

Let \( u = e^x - 2 \), then \( du = e^x \, dx \). Substituting these expressions into the definite integral above we get

\[\int_{\ln 2}^{1} e^x \cos(e^x - 2) \, dx = \int_{\ln 2}^{1-e^2} \cos u \, du \]
\[= \int_{0}^{e^2-2} \cos u \, du \]
\[= \int_{0}^{e^2-2} \cos u \, du \]
\[= \sin u \bigg|_{0}^{e^2-2} \]
\[= \sin 0 - \sin(e - 2) \]
\[= -\sin(e - 2) \]
3. (8 points) Given the graph of $f(x)$ below, find $\int_{-2}^{6} f(x) \, dx$.

Since the definite integral is the net area under a curve, we can use the area formula for a triangle three times to find the definite integral,

$$\int_{-2}^{6} f(x) \, dx = -\left(\frac{1}{2}(2)(2)\right) + \frac{1}{2}(4)(2) + \frac{1}{2}(2)(4) = -2 + 4 + 4 = 6$$
4. (3 points each) Find the following derivatives.

(a) \[ \frac{d}{dx} \int_0^x \sqrt{1 + 2t} \, dt = \sqrt{1 + 2x} \]

(b) \[ \frac{d}{dx} \int_0^1 \frac{t}{1 + t} \, dt = 0 \]

(c) \[ \frac{d}{dx} \int_1^x t^2 \cos t \, dt = -x^2 \cos x \]

(d) \[ \frac{d}{dx} \int_1^\sqrt{x} \frac{\sin t}{t} \, dt = \frac{\sin x}{x} \frac{d}{dx} x^{1/2} = \frac{\sin x}{x} \frac{1}{2} x^{-1/2} = \frac{\sin x}{2x^{3/2}} \]

5. (15 points) Use the limit of a Riemann sum to evaluate the following definite integral.

\[ \int_0^2 2x^2 \, dx \]

Divide the interval \([0, 2]\) into \(n\) equal subintervals of length \(\Delta x = \frac{2}{n}\). These smaller subintervals are \([0, \Delta x], [\Delta x, 2\Delta x], [2\Delta x, 3\Delta x], \ldots, [(i-1)\Delta x, i\Delta x], \ldots, [(n-1)\Delta x, 2]\). Within the \(i\)th subinterval choose \(w_i = i\Delta x = \frac{2i}{n}\). Thus the Riemann sum will have the form

\[ \sum_{i=1}^n \Delta x f(w_i) = \sum_{i=1}^n \frac{2}{n} w_i^2 \]

\[ = \frac{2}{n} \sum_{i=1}^n w_i^2 \]

\[ = \frac{4}{n} \sum_{i=1}^n w_i^2 \]

\[ = \frac{4}{n} \sum_{i=1}^n \left( \frac{2i}{n} \right)^2 \]

\[ = \frac{4}{n} \sum_{i=1}^n \frac{4i^2}{n^2} \]

\[ = \frac{4}{n} \frac{4}{n^2} \sum_{i=1}^n i^2 \]

\[ = \frac{16}{n^3} \sum_{i=1}^n i^2 \]

\[ = \frac{16}{n^3} \frac{n(n+1)(2n+1)}{6} \]

\[ = \frac{4}{3} \frac{(n+1)(2n+1)}{n} \]

\[ = \frac{4}{3} \left( \frac{2n^2 + 3n + 1}{n} \right) \]

\[ = \frac{4}{3} \left( 2n + 3 + \frac{1}{n} \right) \]

\[ = \frac{8n}{3} + 4 + \frac{4}{3n} \]

\[ \lim_{n \to \infty} \left( \frac{8n}{3} + 4 + \frac{4}{3n} \right) = \frac{8}{3} \cdot \lim_{n \to \infty} n + 4 = \frac{8}{3} \cdot \infty + 4 \]

\[ = \infty + 4 = \infty \]

\[ \int_0^2 2x^2 \, dx = \infty \]
Thus the limit of the Riemann sum as \( n \to \infty \) is

\[
\lim_{n \to \infty} R_n = \lim_{n \to \infty} \frac{16n^2 + 24n + 8}{3n^2} = \lim_{n \to \infty} \frac{(16n^2 + 24n + 8) \frac{1}{n^2}}{3n^2 \frac{1}{n^2}} = \lim_{n \to \infty} \frac{16 + \frac{24}{n} + \frac{8}{n^2}}{3} = \frac{16}{3}.
\]

Thus we conclude that

\[
\int_0^2 2x^2 \, dx = \frac{16}{3}.
\]

**Extra Credit** (5 points — No partial credit) If \( f \) is a differentiable function such that

\[
\int_0^x f(t) \, dt = [f(x)]^2 \quad \text{for all } x,
\]

find \( f \).