Alternating Series
MATH 211, *Calculus II*

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Spring 2018
Alternating Series

We have explored **positive term** series and their convergence/divergence properties.

Today we study **alternating series** which have one of two forms:

\[
\sum_{k=1}^{\infty} (-1)^{k+1} a_k = a_1 - a_2 + a_3 - a_4 + \cdots
\]

or

\[
\sum_{k=1}^{\infty} (-1)^k a_k = -a_1 + a_2 - a_3 + a_4 - \cdots
\]

where \( a_k > 0 \) for all \( k = 1, 2, \ldots \).
Examples

\[ \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2k}{k^2 + 2} = \frac{2}{3} - \frac{2}{3} + \frac{6}{11} - \frac{4}{9} + \cdots \]
Examples

\[
\sum_{k=1}^{\infty} (-1)^{k+1} \frac{2k}{k^2 + 2} = \frac{2}{3} - \frac{2}{3} + \frac{6}{11} - \frac{4}{9} + \cdots
\]

\[
\sum_{k=1}^{\infty} (-1)^k \frac{3k}{2 + 4k} = -\frac{1}{2} + \frac{3}{5} - \frac{9}{14} + \frac{2}{3} - \cdots
\]
Alternating Series Test

Theorem (Alternating Series Test)

The alternating series \( \sum_{k=1}^{\infty} (-1)^{k+1} a_k \) is convergent if

1. \( 0 < a_{k+1} \leq a_k \) for all \( k \geq 1 \), and
2. \( \lim_{k \to \infty} a_k = 0 \).

Remarks:
▶ This theorem is not true for positive-term series, e.g., the harmonic series.
▶ Deleting or ignoring a finite number of terms from a series does not alter its convergence or divergence.
Theorem (Alternating Series Test)

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Remarks:

- This theorem is not true for positive-term series, e.g., the harmonic series.
- Deleting or ignoring a finite number of terms from a series does not alter its convergence or divergence.
Proof

Even-indexed partial sums:

\[ S_2 = a_1 - a_2 \geq 0 \]
\[ S_4 = S_2 + a_3 - a_4 \geq S_2 \]

\[ \vdots \]

\[ S_{2n} = S_{2n-2} + a_{2n-1} - a_{2n} \geq S_{2n-2} \]

Conclusion: the sequence of even-indexed partial sums \( \{S_{2n}\}_{n=1}^{\infty} \) is monotonically increasing.

\[ 0 \leq S_{2n} = a_1 + (-a_2 + a_3) + (-a_4 + a_5) + \cdots - a_{2n} \leq a_1 \]

Conclusion: the sequence of even-indexed partial sums \( \{S_{2n}\}_{n=1}^{\infty} \) is bounded, therefore the sequence must converge to a limit \( L \).

\[ S_{2n+1} = S_{2n} + a_{2n+1} \]
\[ \lim_{n \to \infty} S_{2n+1} = \lim_{n \to \infty} S_{2n} + \lim_{n \to \infty} a_{2n+1} = L + 0 = L \]
Examples

Determine if the following alternating series converge or diverge.

1. \[ \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2k}{k^2 + 2} \]
2. \[ \sum_{k=1}^{\infty} (-1)^k \frac{3k}{2 + 4k} \]
3. \[ \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} \quad \text{(the alternating harmonic series)} \]
\[
\sum_{k=1}^{\infty} (-1)^{k+1} \frac{2k}{k^2 + 2}
\]

To use the Alternating Series Test, we must check

1. \(0 < a_{k+1} \leq a_k\),

Let \(f(x) = \frac{2x}{x^2 + 2}\), then \(f'(x) = \frac{4 - x^2}{(x^2 + 2)^2} < 0\) for \(x > 2\), thus the sequence \(\{ a_k \}_{k=1}^{\infty} \) is monotone decreasing for \(k = 2, 3, \ldots\). Since we can ignore a finite number of terms of the infinite series without affecting its convergence or divergence, we can conclude the first hypothesis of the AST holds.

2. \(\lim_{k \to \infty} a_k = 0\),

\[
\lim_{k \to \infty} \frac{2k}{k^2 + 2} = \lim_{k \to \infty} \frac{(2k)/k^2}{(k^2 + 2)/k^2} = \lim_{k \to \infty} \frac{2/k}{1 + 2/k^2} = 0.
\]

Thus the infinite series converges by the Alternating Series Test.
\[
\sum_{k=1}^{\infty} (-1)^k \frac{3k}{2 + 4k}
\]

Note that

\[
\lim_{k \to \infty} \frac{3k}{2 + 4k} = \lim_{k \to \infty} \frac{(3k)/k}{(2 + 4k)/k} = \lim_{k \to \infty} \frac{3}{2/k + 4} = \frac{3}{4} \neq 0
\]

and thus the infinite series diverges by the \textit{kth} Term Test.
\[
\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}
\]

To use the Alternating Series Test, we must check

1. \(0 < a_{k+1} \leq a_k\),

\[
0 < \frac{1}{k+1} \leq \frac{1}{k}
\]
for all \(k = 1, 2, \ldots\).

2. \(\lim_{k \to \infty} a_k = 0\),

\[
\lim_{k \to \infty} \frac{1}{k} = 0
\]

Thus the infinite series converges by the Alternating Series Test.
Theorem

Suppose $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ is a convergent alternating series. If $S$ is the sum of the series and $S_n$ is the $n^{th}$ partial sum of the series, then

$$|S - S_n| \leq a_{n+1}$$

for all $n$. 
Proof

Consider the alternating series $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ and let $n$ be an even integer,

$$
S_n \leq S \leq S_{n+1}
$$

$$
0 \leq S - S_n \leq S_{n+1} - S_n = a_{n+1}
$$

$$
-a_{n+1} \leq S - S_n \leq a_{n+1}
$$

$$
|S - S_n| \leq a_{n+1}.
$$

Similarly we can show that $|S - S_n| \leq a_{n+1}$ when $n$ is odd.
Examples

Determine the smallest value of $n$ sufficient to estimate the sum of the following convergent alternating series to within an error of $10^{-4}$.

1. $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$ (the alternating harmonic series)

2. $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{2k}{k^2 + 2}$

3. $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{3k}{2 + 4k^4}$
We have already shown that this series converges.

\[
|S - S_n| \leq a_{n+1} < 10^{-4}
\]

\[
\frac{1}{n + 1} < 10^{-4}
\]

\[
n + 1 > 10^4
\]

\[
n > 9999 \implies (\text{smallest}) \ n = 10,000
\]
\[ \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} \]

We have already shown that this series converges.

\[ |S - S_n| \leq a_{n+1} < 10^{-4} \]
\[ \frac{1}{n+1} < 10^{-4} \]
\[ n + 1 > 10^4 \]
\[ n > 9999 \implies \text{(smallest)} \ n = 10,000 \]

\[ \sum_{k=1}^{10,000} (-1)^{k+1} \frac{1}{k} \approx 0.693097 \approx \ln 2 \]
\[
\sum_{k=1}^{\infty} (-1)^{k+1} \frac{2k}{k^2 + 2}
\]

We have already shown that this series converges.

\[
|S - S_n| \leq a_{n+1} < 10^{-4}
\]
\[
\frac{2(n + 1)}{(n + 1)^2 + 2} < 10^{-4}
\]
\[
2n + 2 < 10^{-4}(n^2 + 2n + 3)
\]
\[
20,000(n + 1) < n^2 + 2n + 3
\]
\[
0 < n^2 - 19,998n - 19,997
\]
\[
n > 19,999 \implies \text{(smallest) } n = 20,000
\]
\[
\sum_{k=1}^{\infty} (-1)^{k+1} \frac{2k}{k^2 + 2}
\]

We have already shown that this series converges.

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|S - S_n| \leq a_{n+1} < 10^{-4}
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0 < n^2 - 19,998n - 19,997
\]
\[
n > 19,999 \implies (\text{smallest}) \ n = 20,000
\]

\[
\sum_{k=1}^{20,000} (-1)^{k+1} \frac{2k}{k^2 + 2} \approx 0.301765
\]
We can verify via the Alternating Series Test that this infinite series converges.

\[ \sum_{k=1}^{\infty} (-1)^{k+1} \frac{3k}{2 + 4k^4} \]

\[ |S - S_n| \leq a_{n+1} < 10^{-4} \]
\[ \frac{3(n+1)}{2 + 4(n+1)^4} < 10^{-4} \]
\[ n > 18.5743 \implies \text{(smallest)} \; n = 19 \]
\[
\sum_{k=1}^{\infty} (-1)^{k+1} \frac{3k}{2 + 4k^4}
\]

We can verify via the Alternating Series Test that this infinite series converges.

\[
|S - S_n| \leq a_{n+1} < 10^{-4}
\]

\[
\frac{3(n + 1)}{2 + 4(n + 1)^4} < 10^{-4}
\]

\[
n > 18.5743 \implies \text{(smallest)} \quad n = 19
\]

\[
\sum_{k=1}^{19} (-1)^{k+1} \frac{3k}{2 + 4k^4} \approx 0.428897
\]
Comment

Checking the condition mentioned in the Alternating Series Test that $0 < a_{k+1} \leq a_k$ is essential. If this hypothesis is not satisfied, then the series may converge or diverge even if it is alternating and $a_k \to 0$ as $k \to \infty$. 
Example: Convergent

Consider the alternating series,

$$1 - 2 + 1 - \frac{1}{2} + 1 - \frac{1}{4} - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{4} - \frac{1}{8} + \cdots$$

We can see that $a_{k+1} > a_k$ for some terms violating the first condition of the AST. However the sequence of partial sums is

$$\left\{1, -1, 0, \frac{-1}{2}, \frac{1}{2}, 0, \frac{1}{4}, \frac{-1}{4}, 0, \frac{-1}{8}, \frac{1}{8}, 0, \ldots\right\}$$

and thus $S_n \to 0$ as $n \to \infty$. Hence

$$1 - 2 + 1 - \frac{1}{2} + 1 - \frac{1}{4} - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{4} - \frac{1}{8} + \cdots = 0$$
Example: Divergent

Consider the alternating series,

\[ 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{4} + \frac{1}{4} - \frac{1}{8} + \frac{1}{8} - \frac{1}{16} + \frac{1}{16} - \frac{1}{32} + \frac{1}{32} - \frac{1}{64} + \cdots \]

We can see that \( a_{k+1} > a_k \) for some terms violating the first condition of the AST.

The sum of the negative terms is \(-1\) (geometric series) while the positive terms are the Harmonic series (divergent). Hence the original alternating series diverges.
Homework

- Read Section 11.5
- Exercises: WebAssign/D2L