We have developed definite integral formulas for arc length and surface area for curves of the form \( y = f(x) \) with \( a \leq x \leq b \).

\[
\begin{align*}
  s &= \int_a^b \sqrt{1 + (f'(x))^2} \, dx \\
  S &= 2\pi \int_a^b |f(x)| \sqrt{1 + (f'(x))^2} \, dx
\end{align*}
\]

Today we will develop formulas for calculating arc length and surface area for curves described parametrically.
Riemann Sum Approach

Suppose a curve is described by the parametric equations:

\[ x = x(t) \]
\[ y = y(t) \]

where \( a \leq t \leq b \) and \( x'(t) \) and \( y'(t) \) are continuous as well.
Partition \([a, b]\) into \(n\) equal subintervals with \(\Delta t = \frac{b - a}{n}\) and \(t_k = a + k\Delta t\) for \(k = 0, 1, \ldots, n\).

\[
\Delta s_k = \sqrt{(x(t_k) - x(t_{k-1}))^2 + (y(t_k) - y(t_{k-1}))^2}
\]

\[
= \sqrt{(x'(v_k)\Delta t)^2 + (y'(w_k)\Delta t)^2} \quad \text{(by the MVT)}
\]

\[
\approx \sqrt{(x'(w_k))^2 + (y'(w_k))^2\Delta t}
\]
Riemann Sum

Arc length

\[ s \approx \sum_{k=1}^{n} \sqrt{(x'(w_k))^2 + (y'(w_k))^2} \Delta t \]

\[ = \lim_{n \to \infty} \sum_{k=1}^{n} \sqrt{(x'(w_k))^2 + (y'(w_k))^2} \Delta t \]

\[ = \int_{a}^{b} \sqrt{(x'(t))^2 + (y'(t))^2} \, dt \]

Remark: the expression \( ds = \sqrt{(x'(t))^2 + (y'(t))^2} \, dt \) is called differential arc length.
Riemann Sum

Arc length

\[ s \approx \sum_{k=1}^{n} \sqrt{(x'(w_k))^2 + (y'(w_k))^2} \Delta t \]

\[ = \lim_{n \to \infty} \sum_{k=1}^{n} \sqrt{(x'(w_k))^2 + (y'(w_k))^2} \Delta t \]

\[ = \int_{a}^{b} \sqrt{(x'(t))^2 + (y'(t))^2} \, dt \]

Remark: the expression \( ds = \sqrt{(x'(t))^2 + (y'(t))^2} \, dt \) is called differential arc length.
Theorem

For the curve defined parametrically by \( x = x(t), \ y = y(t), \ a \leq t \leq b \), if \( x'(t) \) and \( y'(t) \) are continuous on \([a, b]\) and the curve does not intersect itself (except possibly at a finite number of points), then the arc length \( s \) of the curve is given by

\[
 s = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2} \ dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \ dt.
\]
Find the arc length of the curve given by

\[ x = 2t^2 \]
\[ y = 2t^3 \]

for \( 0 \leq t \leq 2 \).
Solution

\[ s = \int_{0}^{2} \sqrt{(4t)^2 + (6t^2)^2} \, dt \]
\[ = \int_{0}^{2} 2t \sqrt{4 + 9t^2} \, dt \]

Integrate by substitution, letting \( u = 4 + 9t^2 \)
and \( \frac{1}{9} \, du = 2t \, dt \).

\[ s = \int_{4}^{40} \frac{1}{9} u^{1/2} \, du \]
\[ = \left. \frac{2}{27} u^{3/2} \right|_{4}^{40} \]
\[ = \frac{2}{27} (40\sqrt{40} - 8) = \frac{16}{27} (10\sqrt{10} - 1) \]
Example

Find the arc length of the curve given by

\[ x = 3 \sin t \]
\[ y = 3 \cos t - 3 \]

for \( 0 \leq t \leq 2\pi \).
Solution

\[ s = \int_0^{2\pi} \sqrt{(3 \cos t)^2 + (-3 \sin t)^2} \, dt \]

\[ = \int_0^{2\pi} \sqrt{9 \cos^2 t + 9 \sin^2 t} \, dt \]

\[ = \int_0^{2\pi} 3 \, dt \]

\[ = 6\pi \]
Brachistochrone Problem

**Problem:** suppose an object is to slide down a path in the $xy$-plane from the origin to a point below the origin but not on the $y$-axis. Which path should the object follow to accomplish the trip in the least time?
Physical Background

\[
d = |v|t
\]

Conservation of energy:

\[
m g y = \frac{1}{2} m \left( \left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 \right)
\]

- To avoid confusion with “time” use the symbol \( u \) as the parameter of the parametric equations.
- To simplify the mathematics we will assume that \( g y \geq 0 \).
Formula for Trip Time

Suppose the path is parameterized by $x = x(u), y = y(u)$ for $0 \leq u \leq 1$, then

\[ m g y = \frac{1}{2} m \left( \left( \frac{dx}{du} \right)^2 + \left( \frac{dy}{du} \right)^2 \right) \]

\[ 2g y = \left( \frac{dx}{du} \right)^2 + \left( \frac{dy}{du} \right)^2 \]

\[ 2g y = (v(u))^2 \]

\[ \sqrt{2g y} = |v(u)|. \]
Formula for Trip Time

Suppose the path is parameterized by \( x = x(u), y = y(u) \) for \( 0 \leq u \leq 1 \), then

\[
m g y = \frac{1}{2} m \left( \left( \frac{dx}{du} \right)^2 + \left( \frac{dy}{du} \right)^2 \right)
\]

\[
2 g y = \left( \frac{dx}{du} \right)^2 + \left( \frac{dy}{du} \right)^2
\]

\[
2 g y = (v(u))^2
\]

\[
\sqrt{2 g y} = |v(u)|.
\]

Therefore the trip time is

\[
t = \frac{d}{|v|} = \int_0^1 \frac{\sqrt{\left( \frac{dx}{du} \right)^2 + \left( \frac{dy}{du} \right)^2}}{\sqrt{2 g y}} \, du = \frac{1}{\sqrt{2 g}} \int_0^1 \sqrt{\frac{\left( \frac{dx}{du} \right)^2 + \left( \frac{dy}{du} \right)^2}{y}} \, du.
\]
Comparison of Paths

Suppose the object is to move from \((0, 0)\) to \((\pi, 2)\) along a straight line parameterized by

\[
\begin{align*}
x &= \pi u \\
y &= 2u
\end{align*}
\]

for \(0 \leq u \leq 1\). Find the time of travel.
Calculation of Travel Time

\[ t = \frac{1}{\sqrt{2g}} \int_0^1 \sqrt{\frac{(\pi)^2 + (2)^2}{2u}} \, du \]

\[ = \frac{\sqrt{\pi^2 + 4}}{2\sqrt{g}} \int_0^1 u^{-1/2} \, du \]

\[ = \left[ \sqrt{\frac{\pi^2 + 4}{g}} u^{1/2} \right]_0^1 \]

\[ t = \sqrt{\frac{\pi^2 + 4}{g}} \]
Comparison of Paths

Suppose the object is to move from \((0, 0)\) to \((\pi, 2)\) along a cycloid parameterized by

\[
\begin{align*}
    x &= \pi u - \sin \pi u \\
    y &= 1 - \cos \pi u
\end{align*}
\]

for \(0 \leq u \leq 1\). Find the time of travel.
Calculation of Travel Time

\[ t = \frac{1}{\sqrt{2g}} \int_{0}^{1} \sqrt{\frac{(\pi - \pi \cos \pi u)^2 + (\pi \sin \pi u)^2}{1 - \cos \pi u}} \, du \]

\[ = \frac{1}{\sqrt{2g}} \int_{0}^{1} \sqrt{\frac{\pi^2 - 2\pi^2 \cos \pi u + \pi^2 \cos^2 \pi u + \pi^2 \sin^2 \pi u}{1 - \cos \pi u}} \, du \]

\[ = \frac{1}{\sqrt{2g}} \int_{0}^{1} \sqrt{\frac{2\pi^2 - 2\pi^2 \cos \pi u}{1 - \cos \pi u}} \, du \]

\[ = \frac{\pi}{\sqrt{g}} \int_{0}^{1} 1 \, du \]

\[ t = \sqrt{\frac{\pi^2}{g}} \]
Result

\[ \sqrt{\frac{\pi^2}{g}} < \sqrt{\frac{\pi^2 + 4}{g}} \]

Even though the cycloid is a greater *distance* (arc length), it requires less *time*. 
Surface Area

Geometrically we may think of the definite integral for the surface area of a solid of revolution as

\[ S = \int_{a}^{b} 2\pi (\text{radius})(\text{arc length}) \, dx. \]

Thus the surface generated when the parametric curve

\[
\begin{align*}
  x &= x(t) \\
  y &= y(t)
\end{align*}
\]

for \( a \leq t \leq b \) is revolved around the \( x \)-axis has surface area

\[ S = 2\pi \int_{a}^{b} |y(t)| \sqrt{(x'(t))^2 + (y'(t))^2} \, dt. \]
Example

Find the surface area of the solid of revolution generated when the parametric curve

\[
\begin{align*}
  x &= 3t \\
  y &= \sqrt{7}t + 1
\end{align*}
\]

for \(0 \leq t \leq 1\) is revolved around the x-axis.
Solution

\[ S = 2\pi \int_{0}^{1} (\sqrt{7}t + 1)\sqrt{3^2 + (\sqrt{7})^2} \, dt \]

\[ = 2\pi \int_{0}^{1} 4(\sqrt{7}t + 1) \, dt \]

\[ = 8\pi \int_{0}^{1} (\sqrt{7}t + 1) \, dt \]

\[ = \left[ 8\pi \left( \frac{\sqrt{7}}{2}t^2 + t \right) \right]_{0}^{1} \]

\[ = 8\pi \left( \frac{\sqrt{7}}{2} + 1 \right) \]

\[ = 4(\sqrt{7} + 2)\pi \]
Example

Find the surface area of the solid of revolution generated when the parametric curve

\[ x = 4t^2 - 1 \]
\[ y = 3 - 2t \]

for \(-2 \leq t \leq 0\) is revolved around the x-axis.
Solution (1 of 2)

\[
S = 2\pi \int_{-2}^{0} (3 - 2t) \sqrt{(8t)^2 + (-2)^2} \, dt \\
= 2\pi \int_{-2}^{0} (3 - 2t) \sqrt{64t^2 + 4} \, dt \\
= 4\pi \int_{-2}^{0} (3 - 2t) \sqrt{16t^2 + 1} \, dt \\
= 12\pi \int_{-2}^{0} \sqrt{16t^2 + 1} \, dt - 8\pi \int_{-2}^{0} t \sqrt{16t^2 + 1} \, dt
\]

- The first integral can be handled via the trigonometric substitution, \( t = \frac{1}{4} \tan \theta \) and \( dt = \frac{1}{4} \sec^2 \theta \, d\theta \).
- The second integral can be handled via the substitution, \( u = 16t^2 + 1 \) and \( \frac{1}{32} \, du = t \, dt \).
Solution (2 of 2)

\[ S = 12\pi \int_{-2}^{0} \sqrt{16t^2 + 1} \, dt - 8\pi \int_{-2}^{0} t\sqrt{16t^2 + 1} \, dt \]

\[ = 3\pi \int_{-\tan^{-1} 8}^{0} \sec^3 \theta \, d\theta - \frac{\pi}{4} \int_{65}^{1} u^{1/2} \, du \]

\[ = \left[ \frac{3\pi}{2} (\sec \theta \tan \theta - \ln[\sec \theta + \tan \theta]) \right]_{-\tan^{-1} 8}^{0} - \left[ \left( \frac{\pi}{6} u^{3/2} \right) \right]_{65}^{1} \]

\[ = 12\sqrt{65}\pi + \frac{3\pi}{2} \ln(8 + \sqrt{65}) - \frac{\pi}{6} (65\sqrt{65} - 1) \approx 590.89 \]
Homework

- Read Section 9.3
- Exercises: 1–19 odd (arc length), 21–29 odd (surface area)