Page 577, Exercise 32

Evaluate the integral,

\[ \int x^3 \sqrt{x^2 - 1} \, dx. \]

Since the integrand contains an expression of the form \( \sqrt{x^2 - a^2} \) with \( a = 1 \) we will try the trigonometric substitution, \( x = a \sec u = \sec u \) and \( dx = \sec u \tan u \, du \).

\[
\begin{align*}
\int x^3 \sqrt{x^2 - 1} \, dx &= \int \sec^3 u \sqrt{\sec^2 u - 1} \, du \\
&= \int \sec^3 u \tan^2 u \, du \\
&= \int \sec^4 u \tan^2 u \, du \\
&= \int \sec^2 u \tan^2 u \, sec^2 u \, du \\
&= \int (\tan^2 u + 1) \tan^2 u \, sec^2 u \, du \\
&= \int (w^2 + 1) w^2 \, dw \\
&= \frac{1}{5} w^5 + \frac{1}{3} w^3 + C \\
&= \frac{1}{5} \tan^5 u + \frac{1}{3} \tan^3 u + C \\
&= \frac{1}{5} \tan^5 (\sec^{-1} x) + \frac{1}{3} \tan^3 (\sec^{-1} x) + C \\
&= \frac{1}{5} \left[ \tan (\sec^{-1} x) \right]^5 + \frac{1}{3} \left[ \tan (\sec^{-1} x) \right]^3 + C \\
&= \frac{1}{5} \left( \sqrt{x^2 - 1} \right)^5 + \frac{1}{3} \left( \sqrt{x^2 - 1} \right)^3 + C \\
&= \frac{1}{5} (x^2 - 1)^{5/2} + \frac{1}{3} (x^2 - 1)^{3/2} + C
\end{align*}
\]
Find the partial fraction expansion and the antiderivative of the following expression.
\[
\frac{2x^4 + 9x^2 + x - 4}{x^3 + 4x}
\]

Since the degree of the numerator is larger than the degree of the denominator we must first perform polynomial division to write the rational function in the form of a quotient and a remainder.

\[
\frac{2x^4 + 9x^2 + x - 4}{x^3 + 4x} = 2x + \frac{x^2 + x - 4}{x^3 + 4x}
\]

Now we will expand the remainder as the sum of partial fractions. The denominator of the remainder factors as \(x(x^2 + 4)\), a single linear factor and an irreducible quadratic factor.

\[
\frac{x^2 + x - 4}{x^3 + 4x} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}
\]

Letting \(x = 0\) we see that \(A = -1\). Replacing \(A\) in the last equation above produces

\[
\begin{align*}
2x^2 + x & = (Bx + C)x \\
2x^2 + x & = Bx^2 + Cx
\end{align*}
\]

Simply equating the coefficients of the powers of \(x\) on both sides of the last equation yields \(B = 2\) and \(C = 1\). Therefore

\[
\frac{2x^4 + 9x^2 + x - 4}{x^3 + 4x} = 2x - \frac{1}{x} + \frac{2x + 1}{x^2 + 4}
\]

and consequently

\[
\int \frac{2x^4 + 9x^2 + x - 4}{x^3 + 4x} \, dx = \int \left( 2x - \frac{1}{x} + \frac{2x + 1}{x^2 + 4} \right) \, dx
\]

\[
= \int 2x \, dx - \int \frac{1}{x} \, dx + \int \frac{2x + 1}{x^2 + 4} \, dx
\]

\[
= x^2 - \ln|x| + \int \frac{2x}{x^2 + 4} \, dx + \int \frac{1}{x^2 + 4} \, dx
\]

\[
= x^2 - \ln|x| + \ln(x^2 + 4) + \int \frac{1}{x^2 + 4} \, dx
\]

\[
= x^2 - \ln|x| + \ln(x^2 + 4) + \frac{1}{2} \tan^{-1} \left( \frac{x}{2} \right) + C
\]