Page 657, Exercise 36

Determine whether the series converges or diverges.

\[
\sum_{k=0}^{\infty} \frac{k^3 + 2k + 3}{k^4 + 2k^2 + 4}
\]

Suppose we let

\[
a_k = \frac{k^3 + 2k + 3}{k^4 + 2k^2 + 4} \quad \text{and} \quad b_k = \frac{1}{k}
\]

and apply the Limit Comparison Test.

\[
\lim_{k \to \infty} \frac{a_k}{b_k} = \lim_{k \to \infty} \frac{k^3 + 2k + 3}{k^4 + 2k^2 + 4} = \lim_{k \to \infty} \frac{(k^4 + 2k^2 + 3k)}{(k^4 + 2k^2 + 4)} \frac{1}{k^4 + 2k^2 + 4}
\]

\[
= \lim_{k \to \infty} \frac{k^4 + 2k^2 + 3k}{k^4 + 2k^2 + 4} \frac{1 + \frac{2}{k^2} + \frac{3}{k^4}}{1 + \frac{2}{k^2} + \frac{4}{k^4}}
\]

\[
= \lim_{k \to \infty} \frac{1 + 0 + 0}{1 + 0 + 0} = 1 > 0
\]

Since \(\sum_{k=1}^{\infty} b_k\) (the Harmonic Series) diverges, according to the Limit Comparison Test

\[
\sum_{k=0}^{\infty} a_k = a_0 + \sum_{k=1}^{\infty} a_k = \frac{3}{4} + \sum_{k=1}^{\infty} a_k \text{ diverges.}
\]
Determine if the series converges or diverges.

\[ \sum_{k=1}^{\infty} (-1)^{k+1} \frac{3^k}{k} \]

Note the series is an alternating series, so we may apply the Alternating Series Test. One of the conditions which must be verified is the

\[ \lim_{k \to \infty} a_k = \lim_{k \to \infty} \frac{3^k}{k} \quad (\text{indeterminate form } \frac{\infty}{\infty}) \]

\[ = \lim_{k \to \infty} \frac{(\ln 3)3^k}{1} \]

\[ = \lim_{k \to \infty} (\ln 3)3^k \]

\[ = \infty. \]

Thus this series does not satisfy the hypotheses of the Alternating Series Test. Furthermore we see above that it fails the $k^{th}$-term test since the \( \lim_{k \to \infty} a_k \neq 0 \). Consequently we may say that the original infinite series diverges.