Millersville University
Department of Mathematics
MATH 211, *Calculus II*

Please evaluate the following improper integrals.

1. $\displaystyle\int_{-1}^{1} \frac{1}{x^{2/3}}\,dx$

   This integral is improper since the integrand possesses a discontinuity at $x = 0$ and $-1 < 0 < 1$.

$$\begin{aligned}
\int_{-1}^{1} \frac{1}{x^{2/3}}\,dx &= \lim_{M\to 0^-} \int_{-1}^{M} \frac{1}{x^{2/3}}\,dx + \lim_{R\to 0^+} \int_{R}^{1} \frac{1}{x^{2/3}}\,dx \\
&= \lim_{M\to 0^-} \int_{-1}^{M} x^{-2/3}\,dx + \lim_{R\to 0^+} \int_{R}^{1} x^{-2/3}\,dx \\
&= \lim_{M\to 0^-} \left(3x^{1/3}\Big|_{-1}^{M} \right) + \lim_{R\to 0^+} \left(3x^{1/3}\Big|_{R}^{1} \right) \\
&= \lim_{M\to 0^-} \left(3M^{1/3} + 3 \right) + \lim_{R\to 0^+} \left(3 - 3R^{1/3} \right) \\
&= 3 + 3 \\
&= 6
\end{aligned}$$

2. $\displaystyle\int_{-\infty}^{2} \frac{2}{x^2+4}\,dx$

   This integral is improper since the lower limit of integration is $-\infty$.

$$\begin{aligned}
\int_{-\infty}^{2} \frac{2}{x^2+4}\,dx &= \lim_{R\to -\infty} \int_{R}^{2} \frac{2}{x^2+4}\,dx \\
&= \lim_{R\to -\infty} \left(\tan^{-1} \frac{x}{2}\Big|_{R}^{2} \right) \\
&= \lim_{R\to -\infty} \left(\tan^{-1}(1) - \tan^{-1} \frac{R}{2} \right) \\
&= \lim_{R\to -\infty} \left(\frac{\pi}{4} - \tan^{-1} \frac{R}{2} \right) \\
&= \frac{\pi}{4} - \lim_{R\to -\infty} \left(\tan^{-1} \frac{R}{2} \right) \\
&= \frac{\pi}{4} - \left(-\frac{\pi}{2} \right) \\
&= \frac{3\pi}{4}
\end{aligned}$$

3. $\displaystyle\int_{0}^{2} \frac{x+1}{\sqrt{4-x^2}}\,dx$

This integral is improper since the integrand is discontinuous at $x = 2$.

$$\begin{aligned}
\int_0^2 \frac{x+1}{\sqrt{4-x^2}}\,dx &= \lim_{R\to 2^-} \int_0^R \frac{x+1}{\sqrt{4-x^2}}\,dx \\
&= \lim_{R\to 2^-} \int_0^R \left(\frac{x}{\sqrt{4-x^2}} + \frac{1}{\sqrt{4-x^2}} \right) dx \\
&= \lim_{R\to 2^-} \left(-\sqrt{4-x^2} + \sin^{-1} \frac{x}{2} \bigg|_0^R \right) \\
&= \lim_{R\to 2^-} \left(-\sqrt{4-R^2} + 2 + \sin^{-1} \frac{R}{2} \right) \\
&= -2 + 2 + 1 \\
&= 1
\end{aligned}$$

4. $\displaystyle\int_{-1}^{\infty} \frac{1}{x^2+5x+6}\,dx$

This integral is improper since the upper limit is integration is infinite.

$$\begin{aligned}
\int_{-1}^{\infty} \frac{1}{x^2+5x+6}\,dx &= \lim_{R\to\infty} \int_{-1}^R \frac{1}{x^2+5x+6}\,dx \\
&= \lim_{R\to\infty} \int_{-1}^R \frac{1}{(x+2)(x+3)}\,dx
\end{aligned}$$

We must use partial fraction expansion to rewrite the integrand.

$$\begin{aligned}
\frac{1}{(x+2)(x+3)} &= \frac{A}{x+2} + \frac{B}{x+3} \\
&= \frac{A(x+3)+B(x+2)}{(x+2)(x+3)} \\
1 &= A(x+3) + B(x+2)
\end{aligned}$$

When $x = -2$ we see that $A = 1$ and when $x = -3$ we have $B = -1$.

$$\begin{aligned}
\lim_{R\to\infty} \int_{-1}^R \frac{1}{(x+2)(x+3)}\,dx &= \lim_{R\to\infty} \int_{-1}^R \left(\frac{1}{x+2} - \frac{1}{x+3} \right) dx \\
&= \lim_{R\to\infty} \left(\ln|x+2| - \ln|x+3| \Big|_{-1}^R \right) \\
&= \lim_{R\to\infty} \left(\ln|R+2| - \ln|R+3| - \ln 1 + \ln 2 \right) \\
&= \lim_{R\to\infty} \left(\ln \frac{|R+2|}{|R+3|} + \ln 2 \right) \\
&= \lim_{R\to\infty} \left(\ln \left| \frac{R+2}{R+3} \right| + \ln 2 \right) \\
&= \ln 1 + \ln 2 \\
&= \ln 2
\end{aligned}$$

5. $\displaystyle\int_0^1 \frac{4x}{\sqrt{1-x^4}}\,dx$

This integral is improper since the integrand is discontinuous at $x = 1$.

$$\int_0^1 \frac{4x}{\sqrt{1-x^4}}\,dx = \lim_{R\to 1^-}\int_0^R \frac{4x}{\sqrt{1-x^4}}\,dx$$

We will integrate by substitution using $u = x^2$ and $du = 2x\,dx$.

$$\begin{aligned}
\lim_{R\to 1^-}\int_0^R \frac{4x}{\sqrt{1-x^4}}\,dx &= \lim_{R\to 1^-}\int_0^{R^2} \frac{2}{\sqrt{1-u^2}}\,du \\
&= \lim_{R\to 1^-}\left(2\sin^{-1} u\Big|_0^{R^2}\right) \\
&= \lim_{R\to 1^-}\left(2\sin^{-1} R^2\right) \\
&= 2\sin^{-1} 1 \\
&= 2\left(\frac{\pi}{2}\right) \\
&= \pi
\end{aligned}$$

6. $\displaystyle\int_0^2 \frac{1}{\sqrt{|x-1|}}\,dx$

This integral is improper because the integrand is discontinuous at $x = 1$ and $0 < 1 < 2$.

$$\begin{aligned}
\int_0^2 \frac{1}{\sqrt{|x-1|}}\,dx &= \lim_{R\to 1^-}\int_0^R \frac{1}{\sqrt{|x-1|}}\,dx + \lim_{M\to 1^+}\int_M^2 \frac{1}{\sqrt{|x-1|}}\,dx \\
&= \lim_{R\to 1^-}\int_0^R \frac{1}{\sqrt{1-x}}\,dx + \lim_{M\to 1^+}\int_M^2 \frac{1}{\sqrt{x-1}}\,dx \\
&= \lim_{R\to 1^-}\left(-2\sqrt{1-x}\Big|_0^R\right) + \lim_{M\to 1^+}\left(2\sqrt{x-1}\Big|_M^2\right) \\
&= \lim_{R\to 1^-}\left(-2\sqrt{1-R}+2\right) + \lim_{M\to 1^+}\left(2-2\sqrt{M-1}\right) \\
&= 2+2 \\
&= 4
\end{aligned}$$

7. $\displaystyle\int_{-\infty}^{\infty} 2xe^{-x^2}\,dx$

This integral is improper since both limits of integration are infinite.

$$\begin{aligned}
\int_{-\infty}^{\infty} 2xe^{-x^2}\,dx &= \int_{-\infty}^{0} 2xe^{-x^2}\,dx + \int_0^{\infty} 2xe^{-x^2}\,dx \\
&= \lim_{R\to -\infty}\int_R^0 2xe^{-x^2}\,dx + \lim_{M\to\infty}\int_0^M 2xe^{-x^2}\,dx \\
&= \lim_{R\to -\infty}\left(-e^{-x^2}\Big|_R^0\right) + \lim_{M\to\infty}\left(-e^{-x^2}\Big|_0^M\right)
\end{aligned}$$

$$\begin{aligned} &= \lim_{R\to-\infty}\left(-1+e^{-R^2}\right)+\lim_{M\to\infty}\left(-e^{-M^2}+1\right) \\ &= -1+1 \\ &= 0 \end{aligned}$$

8. $\displaystyle\int_0^\infty \frac{16\tan^{-1}x}{1+x^2}\,dx$

   This integral is improper since the upper limit of integration is infinite.

$$\int_0^\infty \frac{16\tan^{-1}x}{1+x^2}\,dx = \lim_{R\to\infty}\int_0^R \frac{16\tan^{-1}x}{1+x^2}\,dx$$

   We will integrate by substitution using

$$u=\tan^{-1}x \quad \text{and} \quad du=\frac{1}{1+x^2}\,dx.$$

$$\begin{aligned} \lim_{R\to\infty}\int_0^R \frac{16\tan^{-1}x}{1+x^2}\,dx &= \lim_{R\to\infty}\int_0^{\tan^{-1}R} 16u\,du \\ &= \lim_{R\to\infty}\left(8u^2\Big|_0^{\tan^{-1}R}\right) \\ &= \lim_{R\to\infty}\left(8\left[\tan^{-1}R\right]^2\right) \\ &= 8\left[\frac{\pi}{2}\right]^2 \\ &= 2\pi^2 \end{aligned}$$

9. $\displaystyle\int_0^\infty 2e^{-x}\sin x\,dx$

   This integral is improper since the upper limit of integration is infinite. In order to find an antiderivative of the integrand, we must use integration by parts, twice. We make the following assignments.

$$\begin{aligned} u &= 2e^{-x} \qquad v=-\cos x \\ du &= -2e^{-x}\,dx \quad dv=\sin x\,dx \end{aligned}$$

   Therefore

$$\int 2e^{-x}\sin x\,dx = -2e^{-x}\cos x-\int 2e^{-x}\cos x\,dx.$$

   For the second integration by parts step we make the assignments:

$$\begin{aligned} u &= 2e^{-x} \qquad v=\sin x \\ du &= -2e^{-x}\,dx \quad dv=\cos x\,dx. \end{aligned}$$

Now we have

$$\begin{aligned}
\int 2e^{-x}\sin x\,dx &= -2e^{-x}\cos x - \int 2e^{-x}\cos x\,dx \\
&= -2e^{-x}\cos x - \left(2e^{-x}\sin x + \int 2e^{-x}\sin x\,dx\right) \\
&= -2e^{-x}\cos x - 2e^{-x}\sin x - \int 2e^{-x}\sin x\,dx \\
2\int 2e^{-x}\sin x\,dx &= -2e^{-x}\cos x - 2e^{-x}\sin x + C \\
\int 2e^{-x}\sin x\,dx &= -e^{-x}\cos x - e^{-x}\sin x + C
\end{aligned}$$

Armed with this antiderivative we may return to the task of evaluating the improper integral.

$$\begin{aligned}
\int_0^\infty 2e^{-x}\sin x\,dx &= \lim_{R\to\infty}\int_0^R 2e^{-x}\sin x\,dx \\
&= \lim_{R\to\infty}\left(-e^{-x}\cos x - e^{-x}\sin x\Big|_0^R\right) \\
&= \lim_{R\to\infty}\left(-e^{-R}\cos R - e^{-R}\sin R + 1\right) \\
&= 1 - \lim_{R\to\infty}\left(e^{-R}\cos R + e^{-R}\sin R\right) \\
&= 1 - \lim_{R\to\infty}\left(\frac{\cos R}{e^R} + \frac{\sin R}{e^R}\right) \\
&= 1 - (0+0) \\
&= 1
\end{aligned}$$

Note that the last limit was evaluated by use of the Squeeze Theorem.

10. $\displaystyle\int_0^1 (-\ln x)\,dx$

This integral is improper since the integrand is discontinuous at $x = 0$.

$$\begin{aligned}
\int_0^1 (-\ln x)\,dx &= \lim_{R\to0^+}\int_R^1 (-\ln x)\,dx \\
&= \lim_{R\to0^+}\left(-x\ln x + x\Big|_R^1\right) \\
&= \lim_{R\to0^+}\left(1 - (-R\ln R + R)\right) \\
&= \lim_{R\to0^+}\left(1 + R\ln R - R)\right) \\
&= 1 + \lim_{R\to0^+}(R\ln R) \\
&= 1 + \lim_{R\to0^+}\frac{\ln R}{\frac{1}{R}}
\end{aligned}$$

$$
\begin{aligned}
&= 1 + \lim_{R \to 0^+} \frac{\frac{1}{R}}{\frac{-1}{R^2}} \\
&= 1 + \lim_{R \to 0^+} (-R) \\
&= 0
\end{aligned}
$$