Integral Test
MATH 211, *Calculus II*

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Remarks:

- Determining the convergence or divergence of a series from its sequence of partial sums is difficult for most series.
- We must develop some indirect techniques for determining if a series converges or diverges.
- Today we will work only with positive term series, i.e., series

\[ \sum_{k=1}^{\infty} a_k \quad \text{where} \quad a_k \geq 0 \quad \text{for all} \; k. \]
Area Under the Curve

Consider \( \sum_{k=1}^{\infty} a_k \) for which there is a function \( f(x) \geq 0 \) for \( x \geq 1 \) and \( f(k) = a_k \) for \( k = 1, 2, \ldots \).

\[ 0 \leq \sum_{k=2}^{n} a_k = S_n - a_1 \leq \int_{1}^{n} f(x) \, dx \]
Boundedness

\[ 0 \leq S_n - a_1 \leq \int_1^n f(x) \, dx \]

\[ a_1 \leq S_n \leq a_1 + \int_1^n f(x) \, dx \]

\[ a_1 \leq S_n \leq a_1 + \int_1^n f(x) \, dx \leq a_1 + \int_1^\infty f(x) \, dx \]

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Remark: the sequence of partial sums is bounded if \( \int_1^\infty f(x) \, dx \) converges.
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Remark: the sequence of partial sums is bounded if \( \int_{1}^{\infty} f(x) \, dx \) converges.
Monotonicity

\[ S_n \leq S_{n+1} \quad \text{for all } n = 1, 2, \ldots \quad \text{Why?} \]
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Why?

\[ \sum_{k=1}^{n} a_k \leq \sum_{k=1}^{n+1} a_k \]
Monotonicity

\[ S_n \leq S_{n+1} \quad \text{for all } n = 1, 2, \ldots \quad \text{Why?} \]

\[ \sum_{k=1}^{n} a_k \leq \sum_{k=1}^{n+1} a_k \]

\[ \sum_{k=1}^{n} a_k \leq a_{n+1} + \sum_{k=1}^{n} a_k \]

Conclusion: if \( \int_{1}^{\infty} f(x) \, dx \) converges, the \( \{S_n\}_{n=1}^{\infty} \) is increasing and bounded and thus must converge. Hence \( \sum_{k=1}^{\infty} a_k \) converges.
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\[ 0 \leq a_{n+1} \]
Monotonicity

\[ S_n \leq S_{n+1} \text{ for all } n = 1, 2, \ldots \] Why?

\[
\sum_{k=1}^{n} a_k \leq \sum_{k=1}^{n+1} a_k \\
\sum_{k=1}^{n} a_k \leq a_{n+1} + \sum_{k=1}^{n} a_k \\
0 \leq a_{n+1}
\]

Conclusion: if \[ \int_{1}^{\infty} f(x) \, dx \] converges the \( \{S_n\}_{n=1}^{\infty} \) is increasing and bounded and thus must converge. Hence \( \sum_{k=1}^{\infty} a_k \) converges.
Area Under the Curve

Consider this scenario:

\[ 0 \leq \int_{1}^{n} f(x) \, dx \leq \sum_{k=1}^{n-1} a_k = S_{n-1} \]
Divergence

\[ \int_1^n f(x) \, dx \leq S_{n-1} \]

\[ \lim_{n \to \infty} \int_1^n f(x) \, dx \leq \lim_{n \to \infty} S_{n-1} \]

\[ \int_1^\infty f(x) \, dx \leq \lim_{n \to \infty} S_{n-1} \]

Remark: if \( \int_1^\infty f(x) \, dx \) diverges, then the sequence of partial sums diverges as well.
Divergence

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Remark: if \( \int_{1}^{\infty} f(x) \, dx \) diverges, then the sequence of partial sums diverges as well.
Theorem (Integral Test)

If $f(k) = a_k$ for all $k = 1, 2, \ldots$, $f$ is continuous and decreasing, and $f(x) \geq 0$ for $x \geq 1$, then the improper integral $\int_1^\infty f(x) \, dx$ and the infinite series $\sum_{k=1}^{\infty} a_k$ either both converge or both diverge.

Remark: when the integral and the series both converge they do not necessarily converge to the same value.
Integral Test

Theorem (Integral Test)

If \( f(k) = a_k \) for all \( k = 1, 2, \ldots \), \( f \) is continuous and decreasing, and \( f(x) \geq 0 \) for \( x \geq 1 \), then the improper integral \( \int_1^\infty f(x) \, dx \) and the infinite series \( \sum_{k=1}^\infty a_k \) either both converge or both diverge.

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Examples

Determine, using the Integral Test, whether the following infinite series converge or diverge.

1. \( \sum_{k=1}^{\infty} \frac{1}{k} \)

2. \( \sum_{k=1}^{\infty} \frac{1}{k^2} \)
Let $f(x) = 1/x$, then $f(k) = 1/k = a_k$ for all $k \in \mathbb{N}$.

\[
\int_1^\infty \frac{1}{x} \, dx = \lim_{R \to \infty} \int_1^R \frac{1}{x} \, dx
\]
\[
= \lim_{R \to \infty} [\ln x]_1^R
\]
\[
= \lim_{R \to \infty} (\ln R - \ln 1) = \infty \quad \text{(diverges)}
\]

Hence the Integral Test shows that the Harmonic Series diverges.
Let \( f(x) = 1/x^2 \), then \( f(k) = 1/k^2 = a_k \) for all \( k \in \mathbb{N} \).

\[
\int_1^\infty \frac{1}{x^2} \, dx = \lim_{R \to \infty} \int_1^R \frac{1}{x^2} \, dx
\]

\[
= \lim_{R \to \infty} \left[ \frac{-1}{x} \right]_1^R
\]

\[
= \lim_{R \to \infty} \left( \frac{-1}{R} + 1 \right) = 1 \text{ (converges)}
\]

Thus by the Integral Test, \( \sum_{k=1}^{\infty} \frac{1}{k^2} \) converges.
$p$-Series

**Definition**

An infinite series of the form $\sum_{k=1}^{\infty} \frac{1}{k^p}$ is called a $p$-series.
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Theorem

The $p$-series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges if $p > 1$ and diverges if $p \leq 1$. 
Suppose \( p = 1 \).

\[
\int_1^\infty \frac{1}{x^p} \, dx = \lim_{R \to \infty} \int_1^R \frac{1}{x} \, dx
\]

\[
= \lim_{R \to \infty} \left[ \ln x \right]_{x=1}^{x=R}
\]

\[
= \lim_{R \to \infty} \ln R = \infty
\]

In this case the improper integral diverges and by the Integral Test, the series \( \sum_{k=1}^{\infty} \frac{1}{k} \) diverges.
Suppose \( p \neq 1 \).

\[
\int_1^\infty \frac{1}{x^p} \, dx = \lim_{R \to \infty} \int_1^R x^{-p} \, dx
\]

\[
= \lim_{R \to \infty} \left[ \frac{1}{1 - p} x^{1-p} \right]_1^R
\]

\[
= \lim_{R \to \infty} \left( \frac{1}{1 - p} \frac{1}{R^{p-1}} - \frac{1}{1 - p} \right)
\]

\[
= \begin{cases} 
\infty & \text{if } p < 1, \\
1/(p - 1) & \text{if } p > 1.
\end{cases}
\]

By the Integral Test, the series \( \sum_{k=1}^\infty \frac{1}{k^p} \) converges if \( p > 1 \).
Examples

Which of the following series converge and which diverge?

1. \( \sum_{k=1}^{\infty} \frac{1}{k^{3/2}} \)
   converges

2. \( \sum_{k=1}^{\infty} \frac{1}{k^{1/4}} \)
   diverges

3. \( \sum_{k=1}^{\infty} \frac{1}{k^{1.001}} \)
   converges

4. \( \sum_{k=1}^{\infty} \frac{1}{k^{-5/4}} \)
   diverges
Examples

Which of the following series converge and which diverge?

1. \[ \sum_{k=1}^{\infty} \frac{1}{k^{3/2}} \] diverges, \( \frac{3}{2} > 1 \)

2. \[ \sum_{k=1}^{\infty} \frac{1}{k^{1/4}} \] diverges

3. \[ \sum_{k=1}^{\infty} \frac{1}{k^{1.001}} \] converges

4. \[ \sum_{k=1}^{\infty} \frac{1}{k^{-5/4}} \] diverges
Which of the following series converge and which diverge?

1. \( \sum_{k=1}^{\infty} \frac{1}{k^{3/2}} \)
   \( \frac{3}{2} > 1 \) converges

2. \( \sum_{k=1}^{\infty} \frac{1}{k^{1/4}} \)
   \( \frac{1}{4} \leq 1 \) diverges

3. \( \sum_{k=1}^{\infty} \frac{1}{k^{1.001}} \)

4. \( \sum_{k=1}^{\infty} \frac{1}{k^{-5/4}} \)
Examples

Which of the following series converge and which diverge?

1. \[ \sum_{k=1}^{\infty} \frac{1}{k^{3/2}} \quad \frac{3}{2} > 1 \text{ converges} \]

2. \[ \sum_{k=1}^{\infty} \frac{1}{k^{1/4}} \quad \frac{1}{4} \leq 1 \text{ diverges} \]

3. \[ \sum_{k=1}^{\infty} \frac{1}{k^{1.001}} \quad 1.001 > 1 \text{ converges} \]

4. \[ \sum_{k=1}^{\infty} \frac{1}{k^{-5/4}} \]
Which of the following series converge and which diverge?

1. \[ \sum_{k=1}^{\infty} \frac{1}{k^{3/2}} \quad \frac{3}{2} > 1 \text{ converges} \]

2. \[ \sum_{k=1}^{\infty} \frac{1}{k^{1/4}} \quad \frac{1}{4} \leq 1 \text{ diverges} \]

3. \[ \sum_{k=1}^{\infty} \frac{1}{k^{1.001}} \quad 1.001 > 1 \text{ converges} \]

4. \[ \sum_{k=1}^{\infty} \frac{1}{k^{-5/4}} \quad -\frac{5}{4} \leq 1 \text{ diverges} \]
Earlier we said that in general $\sum_{k=1}^{\infty} a_k \neq \int_1^{\infty} f(x) \, dx$, but an improper integral can be used to help \textit{estimate} the sum of the series.
Series Remainder

Earlier we said that in general \( \sum_{k=1}^{\infty} a_k \neq \int_{1}^{\infty} f(x) \, dx \), but an improper integral can be used to help \textbf{estimate} the sum of the series.

Let \( S = \sum_{k=1}^{\infty} a_k \) and define the \textbf{remainder} \( R_n \) as

\[
R_n = S - S_n = \sum_{k=1}^{\infty} a_k - \sum_{k=1}^{n} a_k = \sum_{k=n+1}^{\infty} a_k
\]
\[
\sum_{k=n+1}^{\infty} a_k \leq \int_{n}^{\infty} f(x) \, dx
\]
Theorem

Suppose that \( f(k) = a_k \) for all \( k = 1, 2, \ldots \), where \( f \) is continuous and decreasing and \( f(x) \geq 0 \) for all \( x \geq 1 \). Further suppose that \( \int_{1}^{\infty} f(x) \, dx \) converges. The remainder \( R_n \) satisfies the inequality

\[
0 \leq R_n = \sum_{k=n+1}^{\infty} a_k \leq \int_{n}^{\infty} f(x) \, dx.
\]
Example

Estimate the error in using $S_{100}$ to approximate

$$\sum_{k=1}^{\infty} \frac{4}{1 + k^2}.$$
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Estimate the error in using $S_{100}$ to approximate

$$\sum_{k=1}^{\infty} \frac{4}{1 + k^2}.$$ 

$S - S_{100} \leq \int_{100}^{\infty} \frac{4}{1 + x^2} \, dx = \lim_{R \to \infty} \int_{100}^{R} \frac{4}{1 + x^2} \, dx$

$$= \lim_{R \to \infty} \left[ 4 \tan^{-1} x \right]_{x=100}^{x=R} = 4 \lim_{R \to \infty} (\tan^{-1} R - \tan^{-1} 100)$$

$$\approx 0.04$$

**Note:** $S_{100} = \sum_{k=1}^{100} \frac{4}{1 + k^2} \approx 4.2669$
Determine if the following infinite series converge or diverge.

\[
\sum_{k=1}^{\infty} \int_{k}^{k+1} \frac{1}{x^{4/3}} \, dx
\]

\[
\sum_{k=1}^{\infty} \int_{k}^{k+1} x^{1/3} \, dx
\]
\[
\sum_{k=1}^{\infty} \int_{k}^{k+1} \frac{1}{x^{4/3}} \, dx
\]

\[
\sum_{k=1}^{\infty} \int_{k}^{k+1} \frac{1}{x^{4/3}} \, dx = \sum_{k=1}^{\infty} \int_{k}^{k+1} x^{-4/3} \, dx
\]

\[
= \sum_{k=1}^{\infty} \left[ -3x^{-1/3} \right]_{x=k}^{x=k+1}
\]

\[
= 3 \sum_{k=1}^{\infty} \left( \frac{1}{k^{1/3}} - \frac{1}{(k+1)^{1/3}} \right) \quad \text{(telescoping sum)}
\]

\[
= 3
\]
\[ \sum_{k=1}^{\infty} \int_{k}^{k+1} x^{1/3} \, dx \]

\[ \sum_{k=1}^{\infty} \int_{k}^{k+1} x^{1/3} \, dx = \sum_{k=1}^{\infty} \left[ \frac{3}{4} x^{4/3} \right]_{x=k}^{x=k+1} \]

\[ = \frac{3}{4} \sum_{k=1}^{\infty} \left( (k + 1)^{4/3} - k^{4/3} \right) \]

While this sum telescopes, the \( N \)th partial sum is

\[ S_N = \frac{3}{4} \sum_{k=1}^{N} \left( (k + 1)^{4/3} - k^{4/3} \right) = \frac{3}{4} \left( (N + 1)^{4/3} - 1 \right) \]

which diverges as \( N \to \infty \).
Homework

- Read Section 11.3
- Exercises: WebAssign/D2L