

Millersville University
Department of Mathematics
MATH 211, Calculus II
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Please evaluate the following definite and indefinite integrals.

1. $\int_1^e x^3 \ln x \, dx$

Using integration by parts we will let

$$\begin{aligned}u &= \ln x & v &= \frac{1}{4}x^4 \\du &= \frac{1}{x} dx & dv &= x^3 dx\end{aligned}$$

then

$$\begin{aligned}\int_1^e x^3 \ln x \, dx &= \left. \frac{1}{4}x^4 \ln x \right|_1^e - \int_1^e \frac{1}{4}x^3 \, dx \\&= \left. \frac{1}{4}x^4 \ln x \right|_1^e - \left. \frac{1}{16}x^4 \right|_1^e \\&= \left(\frac{1}{4}e^4 \ln e - \frac{1}{4} \ln 1 \right) - \left(\frac{1}{16}e^4 - \frac{1}{16} \right) \\&= \frac{3}{16}e^4 + \frac{1}{16} \\&\approx 10.2997\end{aligned}$$

2. $\int \sin^{-1} x \, dx$

Using integration by parts we will let

$$\begin{aligned}u &= \sin^{-1} x & v &= x \\du &= \frac{1}{\sqrt{1-x^2}} dx & dv &= dx\end{aligned}$$

then

$$\int \sin^{-1} x \, dx = x \sin^{-1} x - \int \frac{1}{\sqrt{1-x^2}} dx.$$

Now we make the substitution $w = 1 - x^2$ and $-\frac{1}{2}dw = x \, dx$ to obtain

$$\begin{aligned}\int \sin^{-1} x \, dx &= x \sin^{-1} x - \int \left(-\frac{1}{2} \right) w^{-1/2} dw \\&= x \sin^{-1} x + \frac{1}{2} \int w^{-1/2} dw \\&= x \sin^{-1} x + w^{1/2} + C \\&= x \sin^{-1} x + \sqrt{1-x^2} + C\end{aligned}$$

3. $\int 4x \sec^2 2x \, dx$

Using integration by parts we will let

$$\begin{aligned} u &= 4x & v &= \frac{1}{2} \tan 2x \\ du &= 4 \, dx & dv &= \sec^2 2x \, dx \end{aligned}$$

then

$$\begin{aligned} \int 4x \sec^2 2x \, dx &= 2x \tan 2x - \int 2 \tan 2x \, dx \\ &= 2x \tan 2x - \int \frac{2 \sin 2x}{\cos 2x} \, dx \end{aligned}$$

Now we make the substitution $w = \cos 2x$ and $-dw = 2 \sin 2x \, dx$ to obtain

$$\begin{aligned} \int 4x \sec^2 2x \, dx &= 2x \tan 2x + \int \frac{1}{w} \, dw \\ &= 2x \tan 2x + \ln |w| + C \\ &= 2x \tan 2x + \ln |\cos 2x| + C \end{aligned}$$

4. $\int p^4 e^{-p} \, dp$

Using the method of tabular integration by parts we obtain

u	dv	sign
	e^{-p}	
p^4	$-e^{-p}$	+
$4p^3$	e^{-p}	-
$12p^2$	$-e^{-p}$	+
$24p$	e^{-p}	-
24	$-e^{-p}$	+
0	e^{-p}	-

and therefore

$$\int p^4 e^{-p} \, dp = -p^4 e^{-p} - 4p^3 e^{-p} - 12p^2 e^{-p} - 24p e^{-p} - 24e^{-p} + C$$

5. $\int_0^{1/\sqrt{2}} 2x \sin^{-1}(x^2) \, dx$

Using integration by substitution we will let $u = x^2$ and $du = 2x \, dx$. Then

$$\int_0^{1/\sqrt{2}} 2x \sin^{-1}(x^2) \, dx = \int_{u(0)}^{u(1/\sqrt{2})} \sin^{-1} u \, du$$

$$\begin{aligned}
&= u \sin^{-1} u + \sqrt{1 - u^2} \Big|_0^{1/2} \quad (\text{using ex. 2}) \\
&= \frac{1}{2} \sin^{-1} \frac{1}{2} + \sqrt{1 - \frac{1}{4}} - 1 \\
&= \frac{1}{2} \cdot \frac{\pi}{6} + \frac{\sqrt{3}}{2} - 1 \\
&= \frac{\pi \sqrt{3} - 2}{12} \\
&\approx 0.127825.
\end{aligned}$$

6. $\int_0^1 x\sqrt{1-x} dx$

We may use integration by parts or integration by substitution for this exercise. First we will tackle it by integration by parts.

$$\begin{aligned}
u &= x & v &= -\frac{2}{3}(1-x)^{3/2} \\
du &= dx & dv &= \sqrt{1-x} dx
\end{aligned}$$

$$\begin{aligned}
\int_0^1 x\sqrt{1-x} dx &= -\frac{2x}{3}(1-x)^{3/2} \Big|_0^1 - \int_0^1 \left(-\frac{2}{3}\right) (1-x)^{3/2} dx \\
&= \frac{2}{3} \int_0^1 (1-x)^{3/2} dx \\
&= -\frac{4}{15}(1-x)^{5/2} \Big|_0^1 \\
&= \frac{4}{15} \\
&\approx 0.266667
\end{aligned}$$

Now we will achieve the same result using integration by substitution. We start by letting

$$\begin{aligned}
u &= 1 - x & \Leftrightarrow & x = 1 - u \\
-du &= dx,
\end{aligned}$$

then

$$\begin{aligned}
\int_0^1 x\sqrt{1-x} dx &= - \int_{u(0)}^{u(1)} (1-u)\sqrt{u} du \\
&= - \int_1^0 (1-u)u^{1/2} du \\
&= \int_0^1 (1-u)u^{1/2} du
\end{aligned}$$

$$\begin{aligned}
&= \int_0^1 u^{1/2} - u^{3/2} du \\
&= \left. \frac{2}{3}u^{3/2} - \frac{2}{5}u^{5/2} \right|_0^1 \\
&= \frac{2}{3} - \frac{2}{5} \\
&= \frac{10}{15} - \frac{6}{15} \\
&= \frac{4}{15} \\
&\approx 0.266667
\end{aligned}$$

7. $\int_0^{\pi/3} x \tan^2 x dx$

We will start this problem by making use of a trigonometric identity, $1 + \tan^2 x = \sec^2 x$.

$$\begin{aligned}
\int_0^{\pi/3} x \tan^2 x dx &= \int_0^{\pi/3} x(-1 + \sec^2 x) dx \\
&= -\int_0^{\pi/3} x dx + \int_0^{\pi/3} x \sec^2 x dx \\
&= -\frac{1}{2}x^2 \Big|_0^{\pi/3} + \int_0^{\pi/3} x \sec^2 x dx \\
&= -\frac{\pi^2}{18} + \int_0^{\pi/3} x \sec^2 x dx
\end{aligned}$$

Now we will use integration by parts on the remaining integral.

$$\begin{aligned}
u &= x & v &= \tan x \\
du &= dx & dv &= \sec^2 x dx
\end{aligned}$$

$$\begin{aligned}
\int_0^{\pi/3} x \tan^2 x dx &= -\frac{\pi^2}{18} + x \tan x \Big|_0^{\pi/3} - \int_0^{\pi/3} \tan x dx \\
&= -\frac{\pi^2}{18} + \frac{\pi}{3}\sqrt{3} - \ln(\cos x) \Big|_0^{\pi/3} \\
&= -\frac{\pi^2}{18} + \frac{\pi\sqrt{3}}{3} - \ln(1/2) \\
&= -\frac{\pi^2}{18} + \frac{\pi\sqrt{3}}{3} + \ln 2 \\
&\approx 0.572341
\end{aligned}$$

8. $\int \sin(\ln x) dx$

Using integration by parts we will make the following assignments.

$$\begin{aligned}
 u &= \sin(\ln x) & v &= x \\
 du &= \cos(\ln x) \cdot \frac{1}{x} dx & dv &= dx
 \end{aligned}$$

Applying the integration by parts formula produces:

$$\int \sin(\ln x) dx = x \sin(\ln x) - \int \cos(\ln x) dx$$

We must use integration by parts again on the remaining integral.

$$\begin{aligned}
 u &= \cos(\ln x) & v &= x \\
 du &= -\sin(\ln x) \cdot \frac{1}{x} dx & dv &= dx
 \end{aligned}$$

Once again the integration by parts formula produces:

$$\begin{aligned}
 \int \sin(\ln x) dx &= x \sin(\ln x) - \left(x \cos(\ln x) + \int \sin(\ln x) dx \right) \\
 &= x \sin(\ln x) - x \cos(\ln x) - \int \sin(\ln x) dx
 \end{aligned}$$

Notice that the remaining integral is the same as the original integral, thus we will add this expression to both sides of the equation.

$$\begin{aligned}
 2 \int \sin(\ln x) dx &= x \sin(\ln x) - x \cos(\ln x) \\
 \int \sin(\ln x) dx &= \frac{1}{2} (x \sin(\ln x) - x \cos(\ln x)) + C
 \end{aligned}$$