Please evaluate the following definite and indefinite integrals.

1. \[ \int_1^e x^3 \ln x \, dx \]

Using integration by parts we will let

\[ u = \ln x \quad v = \frac{1}{4} x^4 \]

\[ du = \frac{1}{x} \, dx \quad dv = x^3 \, dx \]

then

\[
\int_1^e x^3 \ln x \, dx = \frac{1}{4} x^4 \ln x \bigg|_1^e - \int_1^e \frac{1}{4} x^3 \, dx
\]

\[
= \frac{1}{4} x^4 \ln x \bigg|_1^e - \frac{1}{16} x^4 \bigg|_1^e
\]

\[
= \left( \frac{1}{4} e^4 \ln e - \frac{1}{4} \ln 1 \right) - \left( \frac{1}{16} e^4 - \frac{1}{16} \right)
\]

\[
= \frac{3}{16} e^4 + \frac{1}{16}
\]

\[
\approx 10.2997
\]

2. \[ \int \sin^{-1} x \, dx \]

Using integration by parts we will let

\[ u = \sin^{-1} x \quad v = x \]

\[ du = \frac{1}{\sqrt{1-x^2}} \, dx \quad dv = dx \]

then

\[
\int \sin^{-1} x \, dx = x \sin^{-1} x - \int \frac{1}{\sqrt{1-x^2}} \, dx.
\]

Now we make the substitution \( w = 1 - x^2 \) and \( -\frac{1}{2}dw = x \, dx \) to obtain

\[
\int \sin^{-1} x \, dx = x \sin^{-1} x - \int \left( -\frac{1}{2} \right) w^{-1/2} \, dw
\]

\[
= x \sin^{-1} x + \frac{1}{2} \int w^{-1/2} \, dw
\]

\[
= x \sin^{-1} x + \frac{1}{2} \left( w^{1/2} \right) + C
\]

\[
= x \sin^{-1} x + \sqrt{1-x^2} + C
\]
3. \( \int 4x \sec^2 2x \, dx \)

Using integration by parts we will let

\[
\begin{align*}
 u &= 4x \\
 v &= \frac{1}{2} \tan 2x \\
 du &= 4 \, dx \\
 dv &= \sec^2 2x \, dx
\end{align*}
\]

then

\[
\int 4x \sec^2 2x \, dx = 2x \tan 2x - \int 2 \tan 2x \, dx = 2x \tan 2x - \int \frac{2 \sin 2x}{\cos 2x} \, dx
\]

Now we make the substitution \( w = \cos 2x \) and \(-dw = 2 \sin 2x \, dx\) to obtain

\[
\int 4x \sec^2 2x \, dx = 2x \tan 2x + \int \frac{1}{w} \, dw = 2x \tan 2x + \ln |w| + C = 2x \tan 2x + \ln |\cos 2x| + C
\]

4. \( \int p^4 e^{-p} \, dp \)

Using the method of tabular integration by parts we obtain

<table>
<thead>
<tr>
<th>( u )</th>
<th>( dv )</th>
<th>sign</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p^4 )</td>
<td>( e^{-p} )</td>
<td>+</td>
</tr>
<tr>
<td>( 4p^3 )</td>
<td>( -e^{-p} )</td>
<td>-</td>
</tr>
<tr>
<td>( 12p^2 )</td>
<td>( e^{-p} )</td>
<td>+</td>
</tr>
<tr>
<td>( 24p )</td>
<td>( -e^{-p} )</td>
<td>-</td>
</tr>
<tr>
<td>( 24 )</td>
<td>( e^{-p} )</td>
<td>+</td>
</tr>
<tr>
<td>( 0 )</td>
<td>( e^{-p} )</td>
<td>-</td>
</tr>
</tbody>
</table>

and therefore

\[
\int p^4 e^{-p} \, dp = -p^4 e^{-p} - 4p^3 e^{-p} - 12p^2 e^{-p} - 24pe^{-p} - 24e^{-p} + C
\]

5. \( \int_{0}^{\frac{1}{\sqrt{2}}} 2x \sin^{-1}(x^2) \, dx \)

Using integration by substitution we will let \( u = x^2 \) and \( du = 2x \, dx \). Then

\[
\int_{0}^{\frac{1}{\sqrt{2}}} 2x \sin^{-1}(x^2) \, dx = \int_{u(0)}^{u(\frac{1}{\sqrt{2}})} \sin^{-1} u \, du
\]
\[
\begin{align*}
\int_0^1 x \sqrt{1-x} \, dx &= \left. u \sin^{-1} u + \sqrt{1-u^2} \right|_0^1 \\
&= \frac{1}{2} \sin^{-1} \frac{1}{2} + \sqrt{1-\frac{1}{4}} - 1 \\
&= \frac{1}{2} \cdot \frac{\pi}{6} + \frac{\sqrt{3}}{2} - 1 \\
&= \frac{\pi \sqrt{3} - 2}{12} \\
&\approx 0.127825.
\end{align*}
\]

6. \( \int_0^1 x \sqrt{1-x} \, dx \)

We may use integration by parts or integration by substitution for this exercise. First we will tackle it by integration by parts.

\[
\begin{align*}
\int_0^1 x \sqrt{1-x} \, dx &= -\frac{2}{3} (1-x)^{3/2} \bigg|_0^1 - \int_0^1 \left( -\frac{2}{3} \right) (1-x)^{3/2} \, dx \\
&= \frac{2}{3} \int_0^1 (1-x)^{3/2} \, dx \\
&= -\frac{4}{15} (1-x)^{5/2} \bigg|_0^1 \\
&= \frac{4}{15} \\
&\approx 0.266667.
\end{align*}
\]

Now we will achieve the same result using integration by substitution. We start by letting

\[
\begin{align*}
u &= 1-x \quad \Leftrightarrow \quad x = 1-u \\
-du &= dx,
\end{align*}
\]

then

\[
\begin{align*}
\int_0^1 x \sqrt{1-x} \, dx &= -\int_{u(0)}^{u(1)} (1-u)^{3/2} \, du \\
&= -\int_0^1 (1-u)^{1/2} \, du \\
&= \int_0^1 (1-u)^{1/2} \, du
\end{align*}
\]
\[ \int_0^1 u^{1/2} - u^{3/2} \, du = \left[ \frac{2}{3} u^{3/2} - \frac{2}{5} u^{5/2} \right]_0^1 \]
\[ = \frac{2}{3} - \frac{2}{5} \]
\[ = \frac{10}{15} - \frac{6}{15} \]
\[ = \frac{4}{15} \approx 0.26667 \]

7. \( \int_{0}^{\pi/3} x \tan^2 x \, dx \)

We will start this problem by making use of a trigonometric identity, \( 1 + \tan^2 x = \sec^2 x \).
\[ \int_{0}^{\pi/3} x \tan^2 x \, dx = \int_{0}^{\pi/3} x(-1 + \sec^2 x) \, dx \]
\[ = -\int_{0}^{\pi/3} x \, dx + \int_{0}^{\pi/3} x \sec^2 x \, dx \]
\[ = -\frac{1}{2} x^2|_{0}^{\pi/3} + \int_{0}^{\pi/3} x \sec^2 x \, dx \]
\[ = -\frac{\pi^2}{18} + \int_{0}^{\pi/3} x \sec^2 x \, dx \]

Now we will use integration by parts on the remaining integral.
\[ u = x \quad v = \tan x \]
\[ du = dx \quad dv = \sec^2 x \, dx \]
\[ \int_{0}^{\pi/3} x \tan^2 x \, dx = -\frac{\pi^2}{18} + x \tan x|_{0}^{\pi/3} - \int_{0}^{\pi/3} \tan x \, dx \]
\[ = -\frac{\pi^2}{18} + \frac{\pi}{3} \sqrt{3} - \ln(\cos x)|_{0}^{\pi/3} \]
\[ = -\frac{\pi^2}{18} + \frac{\pi}{3} \sqrt{3} - \ln(1/2) \]
\[ = -\frac{\pi^2}{18} + \frac{\pi}{3} \sqrt{3} + \ln 2 \]
\[ \approx 0.572341 \]

8. \( \int \sin(\ln x) \, dx \)

Using integration by parts we will make the following assignments.
\[ u = \sin(\ln x) \quad \quad \quad v = x \]
\[ du = \cos(\ln x) \cdot \frac{1}{x} \, dx \quad dv = dx \]

Applying the integration by parts formula produces:

\[
\int \sin(\ln x) \, dx = x \sin(\ln x) - \int \cos(\ln x) \, dx
\]

We must use integration by parts again on the remaining integral.

\[ u = \cos(\ln x) \quad \quad \quad v = x \]
\[ du = -\sin(\ln x) \cdot \frac{1}{x} \, dx \quad dv = dx \quad \text{Once again the integration by parts formula produces:} \]

\[
\int \sin(\ln x) \, dx = x \sin(\ln x) - \left( x \cos(\ln x) + \int \sin(\ln x) \, dx \right)
\]
\[
= x \sin(\ln x) - x \cos(\ln x) - \int \sin(\ln x) \, dx
\]

Notice that the remaining integral is the same as the original integral, thus we will add this expression to both sides of the equation.

\[
2 \int \sin(\ln x) \, dx = x \sin(\ln x) - x \cos(\ln x)
\]
\[
\int \sin(\ln x) \, dx = \frac{1}{2} \left( x \sin(\ln x) - x \cos(\ln x) \right) + C
\]