Volume: Method of Cylindrical Shells
MATH 211, Calculus II

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Yesterday we used **disks** and **washers** to find the volume of a solid of revolution.
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For some types of solid regions decomposing the volume into **cylindrical shells** may be more convenient.
Cylindrical Shells

Consider a cylindrical shell of **height** $h$, **inner radius** $r_1$, and **outer radius** $r_2$. 
Volume of a Cylindrical Shell

\[ \Delta V = \pi r_2^2 h - \pi r_1^2 h = \pi h(r_2^2 - r_1^2) \]
\[ = \pi h(r_2 + r_1)(r_2 - r_1) = \pi h(r_1 + r_2)\Delta r \]
\[ = 2\pi h \frac{r_1 + r_2}{2} \Delta r = 2\pi r h \Delta r \]

where \( \Delta r = r_2 - r_1 \) is the **thickness** of the shell and \( r = \frac{r_1 + r_2}{2} \)

is the **average radius** of the shell.
Imagine that the shaded region and the rectangle pictured below are rotated completely around the $y$-axis.
Solid (Different Perspective)
Solid (Different Perspective)
Solid (Different Perspective)
Solid (Different Perspective)
Solid (Different Perspective)
Solid (Different Perspective)
Riemann Sum Approach

Suppose \( f \) is continuous on \([a, b]\) and \( f(x) \geq 0 \) for \( a \leq x \leq b \) and the region bounded below the graph of \( y = f(x) \), above the \( x \)-axis and between \( x = a \) and \( x = b \) is revolved around the \( y \)-axis.

Let \( \Delta x = (b - a)/n \) for some \( n \in \mathbb{N} \) and \( x_i = a + \left( i - \frac{1}{2} \right) \Delta x \) for \( i = 1, 2, \ldots, n \), then

\[
V \approx \sum_{i=1}^{n} 2\pi x_i f(x_i) \Delta x
\]

\[
V = \lim_{n \to \infty} \sum_{i=1}^{n} 2\pi x_i f(x_i) \Delta x
\]

\[
V = \int_{a}^{b} 2\pi \underbrace{x}_{(radius)} \underbrace{f(x)}_{(height)} \underbrace{dx}_{(thickness)}
\]
Example

Find the volume of the solid of revolution generated when the region bounded between $y = 1/x$, the $x$-axis, and the lines $x = 1$ and $x = 2$ is revolved around the $y$-axis.
Solution

\[ V = 2\pi \int_1^2 x \left( \frac{1}{x} \right) \, dx \]

\[ = 2\pi \int_1^2 1 \, dx \]

\[ = [2\pi x]_{x=1}^{x=2} \]

\[ = 2\pi (2 - 1) \]

\[ V = 2\pi \]
Example

Find the volume of the solid of revolution generated when the region bounded between \( y = \sqrt[3]{x} \), the \( x \)-axis, and the lines \( x = 1 \) and \( x = 8 \) is revolved around the \( y \)-axis.
Solution

\[ V = 2\pi \int_1^8 x \left( \frac{3}{\sqrt[3]{x}} \right) \, dx \]

\[ = 2\pi \int_1^8 x^{4/3} \, dx \]

\[ = \left[ 2\pi \left( \frac{3}{7} \right) x^{7/3} \right]_{x=1}^{x=8} \]

\[ = \frac{6}{7} \pi \left( 8^{7/3} - 1^{7/3} \right) \]

\[ = \frac{6}{7} \pi \left( 128 - 1 \right) \]

\[ V = \frac{762}{7} \pi \]
Revolving Around the \( x \)-axis

Find the volume of the solid of revolution generated when the region bounded between the graph of \( y^3 = x \), the \( y \)-axis, and the line \( y = 3 \) is revolved around the \( x \)-axis.
Solution

\[ V = 2\pi \int_{0}^{3} y \left(y^3\right) \, dy \]

\[ = 2\pi \int_{0}^{3} y^4 \, dy \]

\[ = \left[ 2\pi \left(\frac{1}{5}\right) y^5 \right]_{y=0}^{y=3} \]

\[ = \frac{2}{5} \pi (3^5 - 0^5) \]

\[ = \frac{2}{5} \pi (243) \]

\[ V = \frac{486}{5} \pi \]
Revolving Around Other Lines

Find the volume of the solid of revolution generated when the region bounded between the graphs of \( y = x^{2/3} \) and \( y = x^2 \) is revolved around the line \( y = -1 \).
Solution

\[ V = 2\pi \int_0^1 (y + 1)(y^{1/2} - y^{3/2}) \, dy \]
\[ = 2\pi \int_0^1 (y^{3/2} - y^{5/2} + y^{1/2} - y^{3/2}) \, dy \]
\[ = 2\pi \int_0^1 (y^{1/2} - y^{5/2}) \, dy \]
\[ = \left[ 2\pi \left( \frac{2}{3} y^{3/2} - \frac{2}{7} y^{7/2} \right) \right]_{y=0}^{y=1} \]
\[ = 2\pi \left( \frac{2}{3} - \frac{2}{7} \right) \]
\[ = 2\pi \left( \frac{14 - 6}{21} \right) \]
\[ V = \frac{16\pi}{21} \]
Example

Find the volume of the solid of revolution generated when the region bounded between the graphs of $x = 2y - y^2$ and $x + y = 0$ is revolved around the line $y = -1$. 
Solution

\[ V = 2\pi \int_0^3 (y + 1)(2y - y^2 - [-y]) \, dy \]

\[ = 2\pi \int_0^3 (y + 1)(3y - y^2) \, dy \]

\[ = 2\pi \int_0^3 (2y^2 - y^3 + 3y) \, dy \]

\[ = \left[ 2\pi \left( \frac{2}{3}y^3 - \frac{1}{4}y^4 + \frac{3}{2}y^2 \right) \right]_{y=0}^{y=3} \]

\[ = 2\pi \left( 18 - \frac{81}{4} + \frac{27}{2} \right) \]

\[ = 2\pi \left( \frac{72 - 81 + 54}{4} \right) \]

\[ = \frac{45\pi}{2} \]
Summary

- Sketch the region to be revolved.
- Determine the variable of integration \((x \ vs. \ y)\).
- Based on the variable of integration and the axis of revolution determine whether to use the method of disks or the method of shells.
- Label the inner and outer radii for washers or the radius and height for shells.
- Evaluate the appropriate integrals.
Example

The region bounded by $y = -x^2 + 6x - 8$ and $y = 0$ is rotated about the $y$-axis. Find the volume of the resulting solid by any method.
Example

The region bounded by $y = -x^2 + 6x - 8$ and $y = 0$ is rotated about the $y$-axis. Find the volume of the resulting solid by any method.
Solid of Revolution
Volume: Method of Shells

\[
V = 2\pi \int_2^4 x(-x^2 + 6x - 8) \, dx
\]
\[
= 2\pi \int_2^4 (-x^3 + 6x^2 - 8x) \, dx
\]
\[
= 2\pi \left[ \left. \frac{-x^4}{4} + 2x^3 - 4x^2 \right| \right]_{x=2}^{x=4}
\]
\[
= 2\pi \left( \frac{-4^4}{4} + 2(4)^3 - 4(4)^2 \right) - 2\pi \left( \frac{-2^4}{4} + 2(2)^3 - 4(2)^2 \right)
\]
\[
= 8\pi
\]
Example

The region bounded by $x = 2 - y^2$ and $y = \sqrt[4]{x}$ is rotated about the line $y$-axis. Find the volume of the resulting solid by any method.
Example

The region bounded by $x = 2 - y^2$ and $y = \sqrt[4]{x}$ is rotated about the line $y$-axis. Find the volume of the resulting solid by any method.
Solid of Revolution
Volume: Method of Disks

\[ V = \pi \int_0^1 (2 - y^2)^2 - (y^4)^2 \, dy \]

\[ = \pi \int_0^1 (4 - 4y^2 + y^4 - y^8) \, dy \]

\[ = \pi \left[ 4y - \frac{4y^3}{3} + \frac{y^5}{5} - \frac{y^9}{9} \right]_y^1 \]

\[ = \pi \left( 4 - \frac{4}{3} + \frac{1}{5} - \frac{1}{9} \right) \]

\[ = \frac{124\pi}{45} \]
Homework

- Read Section 6.3
- Exercises: see WebAssign/D2L