Applications of Integration to Physics and Engineering
MATH 211, *Calculus II*

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Spring 2011
**mass**: quantity of matter (units: kg or g (metric) or slugs (English))

**weight**: a force related to mass through Newton’s Second Law

\[
\text{weight} = (\text{mass})(\text{gravitational acceleration})
\]

**gravity**: gravitational acceleration (notation, \(g\))

- Metric units \(g = 9.8 \text{ m/s}^2\) or \(g = 980 \text{ cm/s}^2\).
- English units, \(g = 32 \text{ ft/s}^2\).
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Units of work:

<table>
<thead>
<tr>
<th>force</th>
<th>distance</th>
<th>work</th>
</tr>
</thead>
<tbody>
<tr>
<td>pound (lb)</td>
<td>foot (ft)</td>
<td>foot-pound (ft-lb)</td>
</tr>
<tr>
<td>inch (in)</td>
<td>inch-pound (in-lb)</td>
<td></td>
</tr>
<tr>
<td>Newton (N)</td>
<td>meter (m)</td>
<td>Newton-meter (N-m)</td>
</tr>
</tbody>
</table>

**Note:** 1 Newton-meter = 1 Joule = 1 J.
Suppose the force \( f \) depends on position \( x \), i.e., the force \( f(x) \) is moved in a straight line from \( x = a \) to \( x = b \).
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Let $n \in \mathbb{N}$ and define $\Delta x = \frac{b-a}{n}$ and $x_i = a + i \Delta x$ for $i = 1, 2, \ldots, n$.

The work done moving the force from $x_{k-1}$ to $x_k$ is approximately $\Delta W_k = f(x_k) \Delta x$. 
Variable Forces

Suppose the force $f$ depends on position $x$, i.e., the force $f(x)$ is moved in a straight line from $x = a$ to $x = b$.

Let $n \in \mathbb{N}$ and define $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$ for $i = 1, 2, \ldots, n$.

The work done moving the force from $x_{k-1}$ to $x_k$ is approximately $\Delta W_k = f(x_k)\Delta x$.

A Riemann sum for the work is then

$$W \approx \sum_{k=1}^{n} f(x_k)\Delta x$$

$$= \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k)\Delta x$$

$$= \int_{a}^{b} f(x) \, dx.$$
**Hooke’s Law:** the force required to stretch or compress a spring beyond its natural length is $f(x) = kx$ where $k$ is called the *spring constant*. 
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**Example**

A force of 10 lb is required to stretch a spring from its natural length of 7 in to 7.5 in. Find the work done in stretching the spring. How much work is done stretching the spring from 7.5 in to 9 in?
Solution

- Since $10 = k(7.5 - 7)$ then the spring constant $k = 20$.
- The work done to stretch the spring the initial 1/2 inch is
  \[ W = \int_0^{1/2} 20x \, dx = 10x^2 \bigg|_0^{1/2} = \frac{5}{2} \text{ in-lb.} \]
- The work done to stretch the spring the next 1.5 inches is
  \[ W = \int_{1/2}^{2} 20x \, dx = 10x^2 \bigg|_{1/2}^{2} = \frac{75}{2} \text{ in-lb.} \]
Example

A hemispherical tank of radius 5 m is full of water with a density of $10^3$ kg/m$^3$. Find the work done in pumping the water out of the tank.
Imagine a thin layer of water parallel to the surface of the tank at a depth of $y$.

The volume of the thin layer of water can be expressed as

$$\pi (25 - y^2) \, dy.$$ 

The weight of the thin layer of water is therefore

$$1000(9.8)\pi (25 - y^2) \, dy = 9800\pi (25 - y^2) \, dy.$$ 

The distance this thin layer must be lifted to remove it from the tank is $0 - y = -y$. 
The work done is

\[
W = \int_{-5}^{0} 9800\pi (25 - y^2)(-y) \, dy
\]

\[
= 9800\pi \int_{-5}^{0} (y^3 - 25y) \, dy
\]

\[
= 9800\pi \left( \frac{1}{4}y^4 - \frac{25}{2}y^2 \right) \bigg|_{-5}^{0}
\]

\[
= -9800\pi \left( \frac{1}{4}(-5)^4 - \frac{25}{2}(-5)^2 \right)
\]

\[
= 1531250\pi \text{ J.}
\]
Example

A 50-ft cable weighing a total of 25 lbs is attached to a 600 lb object. Find the work done in using the cable to lift the object 30 ft.
Solution

- The cable weighs $1/2 \text{ lb/ft}$.
- When the object has been lifted $x$ feet, the total weight of the object and the remaining length of cable is $f(x) = 600 + 25 - x/2 = 625 - x/2$.
- The work done is

$$W = \int_{0}^{30} \left(625 - \frac{x}{2}\right) \, dx = \left(625x - \frac{x^2}{4}\right)\bigg|_{0}^{30} = 18525 \text{ ft-lb}.$$
In mathematics, the physical sciences, and engineering it is convenient to replace a rigid object of mass $m$ by an idealized point-mass (also of $m$).
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**Question:** where do we place the mass?
Suppose two masses $m_1$ and $m_2$ are attached to opposite ends of a rod.

Where can we support the rod so that the system is balanced?

Let $\bar{x}$ be the location of the balance point.
Example (2 of 2)

\[ m_1(\bar{x} - x_1) = m_2(x_2 - \bar{x}) \]
\[ \bar{x}(m_1 + m_2) = m_1 x_1 + m_2 x_2 \]
\[ \bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \]

Remark: the coordinate \( \bar{x} \) is called the center of mass (or center of gravity) of the system.
General Case

Definition

If $S$ denotes a set of point masses $\{m_1, m_2, \ldots, m_n\}$ located at the points $\{x_1, x_2, \ldots, x_n\}$ respectively along the $x$-axis, then

- the **total mass** is $m = \sum_{i=1}^{n} m_i$,

- the **moment about the origin** is $M_0 = \sum_{i=1}^{n} m_i x_i$,

- the **center of mass** is $\bar{x} = \frac{M_0}{m}$.
Example

Suppose $S$ consists of masses $\{5, 7, 11, 13\}$ kg located at $\{-3, -1, 1, 2\}$ respectively along the $x$-axis. Find the center of mass of $S$. 
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m = 5 + 7 + 11 + 13 = 36
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\[
M_0 = (5)(-3) + (7)(-1) + (11)(1) + (13)(2) = 15
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Suppose $S$ consists of masses $\{5, 7, 11, 13\}$ kg located at $\{-3, -1, 1, 2\}$ respectively along the $x$-axis. Find the center of mass of $S$.

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\begin{align*}
m &= 5 + 7 + 11 + 13 = 36 \\
M_0 &= (5)(-3) + (7)(-1) + (11)(1) + (13)(2) = 15 \\
\overline{x} &= \frac{15}{36} = \frac{5}{12}
\end{align*}
\]
Distributed Case

Suppose an object is continuously distributed along the $x$-axis in the interval $[a, b]$ and the **density** (mass/length) of the object is given by $\rho(x)$. 


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Suppose an object is continuously distributed along the $x$-axis in the interval $[a, b]$ and the density (mass/length) of the object is given by $\rho(x)$.

Question: how can we find the center of mass of such an object?

Answer: a Riemann Sum! Let $n \in \mathbb{N}$ and define $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$. The mass of the portion of the object in the interval $[x_{k-1}, x_k]$ is $\Delta m_k \approx \rho(x_k)\Delta x$. Thus the total mass of the object is

$$m \approx \sum_{k=1}^{n} \rho(x_k)\Delta x$$

$$= \lim_{n \to \infty} \sum_{k=1}^{n} \rho(x_k)\Delta x = \int_{a}^{b} \rho(x) \, dx.$$
We can use a Riemann sum to find the moment about the origin of the distributed object.

\[ M_0 \approx \sum_{k=1}^{n} x_k \rho(x_k) \Delta x \]

\[ = \lim_{n \to \infty} \sum_{k=1}^{n} x_k \rho(x_k) \Delta x \]

\[ = \int_{a}^{b} x \rho(x) \, dx \]

Thus the center of mass is

\[ \overline{x} = \frac{M_0}{m} = \frac{\int_{a}^{b} x \rho(x) \, dx}{\int_{a}^{b} \rho(x) \, dx}. \]
Example

Find the mass and center of mass of an object whose density is given by the \( \rho(x) = \frac{x}{5} + 1 \) for \( 0 \leq x \leq 7 \).
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m = \int_0^7 \left( \frac{x}{5} + 1 \right) \, dx = \left( \frac{x^2}{10} + x \right) \bigg|_0^7 = \frac{119}{10}
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M_0 = \int_0^7 \left( \frac{x^2}{5} + x \right) \, dx = \left. \left( \frac{x^3}{15} + \frac{x^2}{2} \right) \right|_0^7 = \frac{1421}{30}
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\]

\[
\bar{x} = \frac{1421}{30} \cdot \frac{119}{10} = 203 \cdot \frac{51}{51}
\]
Example

Find the mass and center of mass of an object whose density is given by the \( \rho(x) = -x^2 - x + 6 \) for \(-3 \leq x \leq 2\).
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\[
\begin{align*}
m &= \int_{-3}^{2} \left( -x^2 - x + 6 \right) \, dx = \left( -\frac{x^3}{3} - \frac{x^2}{2} + 6x \right) \bigg|_{-3}^{2} = \frac{125}{6} \\
M_0 &= \int_{-3}^{2} \left( -x^3 - x^2 + 6x \right) \, dx = \left( -\frac{x^4}{4} - \frac{x^3}{3} + 3x^2 \right) \bigg|_{-3}^{2} = -\frac{125}{12} \\
\bar{x} &= -\frac{\frac{125}{12}}{\frac{125}{6}} = -\frac{1}{2}
\end{align*}
\]
Terminology:

**pressure:** force exerted per unit area (notation, \( p \))
- **gravity:** gravitational acceleration (notation, \( g \))
  - Metric units \( g = 9.8 \text{ m/s}^2 \) or \( g = 980 \text{ cm/s}^2 \).
  - English units, \( g = 32 \text{ ft/s}^2 \).

**density:** mass per unit volume (notation, \( \rho \))
- for water \( \rho = 1000 \text{ kg/m}^3 \) or \( \rho = 1 \text{ g/cm}^3 \).
- English units, \( \rho g = 62.4 \text{ lb/ft}^3 \).

**depth:** distance to the surface of a fluid (notation, \( h \))
Pascal’s Principle: the pressure exerted at a depth $h$ in a fluid is the same in every direction.

If the area of a plate is $A$ then the force on the plate is $\rho ghA$, provided the plate is entirely at depth $h$. 
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If the area of a plate is $A$ then the force on the plate is $\rho ghA$, provided the plate is entirely at depth $h$.

Question: what if the plate is not oriented horizontally?
Suppose the plate is oriented vertically (parallel to the \(xy\)-plane) so that the width of the plate is a function of \(y\), call it \(w(y)\).

Suppose the plate lies in the interval (on the \(y\)-axis) \([a, b]\).

Let \(n \in \mathbb{N}\) and \(\Delta y = \frac{b-a}{n}\) and \(y_i = a + i\Delta y\), then the force on the portion of the plate between \(y_{k-1}\) and \(y_k\) is

\[
\Delta F_k \approx \rho gh(y_k)w(y_k)\Delta y.
\]

Total hydrostatic force is

\[
F \approx \sum_{k=1}^{n} \rho gh(y_k)w(y_k)\Delta y
\]

\[
= \lim_{n \to \infty} \sum_{k=1}^{n} \rho gh(y_k)w(y_k)\Delta y
\]

\[
= \rho g \int_{a}^{b} h(y)w(y) \, dy
\]
Example

A dam has a submerged gate in the shape of an equilateral triangle, two feet on a side with the horizontal base nearest the surface of the water and ten feet below it. Find the force on the gate.
Solution

- The edges of the plate are described by the lines with equations $y = -2 + 2x$ and $y = -2 - 2x$.
- The width of the plate is $w(y) = 2 + y$
- The hydrostatic force is

$$F = 62.4 \int_{-2}^{0} (10 - y)(2 + y) \, dy$$

$$= 62.4 \int_{-2}^{0} (20 + 8y - y^2) \, dy$$

$$= 62.4 \left( \frac{64}{3} \right)$$

$$\approx 1331.2 \text{ lb.}$$
Example

A square plate of sides 5 feet is submerged vertically in water such that one of the diagonals is parallel to the surface of the water. If the distance from the surface of the water to the center of the plate is 4 feet, find the force exerted by the water on one side of the plate.
The edges of the plate are described by the lines with equations $y - x = 5/\sqrt{2}$ and $x + y = 5/\sqrt{2}$ (for $y \geq 0$) and by $x + y = -5/\sqrt{2}$ and $y - x = -5/\sqrt{2}$ (for $y < 0$).

The width of the plate is

$$w(y) = \begin{cases} 
\frac{10}{\sqrt{2}} - 2y & \text{if } y \geq 0, \\
\frac{10}{\sqrt{2}} + 2y & \text{if } y < 0.
\end{cases}$$

The hydrostatic force is

$$F = 62.4 \int_0^{5/\sqrt{2}} (4 - y) \left( \frac{10}{\sqrt{2}} - 2y \right) \, dy + 62.4 \int_{-5/\sqrt{2}}^0 (4 - y) \left( \frac{10}{\sqrt{2}} + 2y \right) \, dy$$

$$= 62.4 \left( 50 - \frac{125}{6\sqrt{2}} \right) + 62.4 \left( 50 + \frac{125}{6\sqrt{2}} \right) = 6240 \text{ lb.}$$
Homework

- Read Section 5.6
- Exercises: 1–13 (work), 21–30 (center of mass), 35–40 (hydrostatic force)