l’Hôpital’s Rule
MATH 211, *Calculus II*

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Limits like the following two can be evaluated using clever algebraic and trigonometric arguments.

1. \( \lim_{x \to 3} \frac{x^2 - x - 6}{x^2 - 9} \)

2. \( \lim_{x \to 0} \frac{\cos x - 1}{x} \)

Today we explore a means of evaluating these and other limits more conveniently.
Indeterminate Forms

Definition
A limit of the form
\[
\lim_{x \to c} \frac{f(x)}{g(x)}
\]
where \( \lim_{x \to c} f(x) = 0 \) and \( \lim_{x \to c} g(x) = 0 \) is said to be indeterminate of the form \( 0/0 \).

Definition
A limit of the form
\[
\lim_{x \to c} \frac{f(x)}{g(x)}
\]
where \( \lim_{x \to c} f(x) = \pm \infty \) and \( \lim_{x \to c} g(x) = \pm \infty \) is said to be indeterminate of the form \( \infty/\infty \).
l’Hôpital’s Rule

**Theorem**

Suppose that $f$ and $g$ are differentiable on the interval $(a, b)$, except possibly at some fixed point $c \in (a, b)$ and that $g'(x) \neq 0$ on $(a, b)$, again except possibly at $x = c$. Suppose further that

\[ \lim_{x \to c} \frac{f(x)}{g(x)} \]

is indeterminate of the form $0/0$ or $\infty/\infty$ and that

\[ \lim_{x \to c} \frac{f'(x)}{g'(x)} = L \text{ (or } \pm \infty). \]

Then

\[ \lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}. \]
Examples

Example

Use l’Hôpital’s Rule as appropriate to evaluate the following limits.

1. \( \lim_{{x \to 0}} \frac{\cos x - 1}{x} \)

2. \( \lim_{{x \to \pi/2}} \frac{1 - \sin x}{2 \cos x} \)

3. \( \lim_{{x \to 0}} \frac{e^x - e^{-x} - 2 \sin x}{x \sin x} \)

4. \( \lim_{{x \to \infty}} \frac{x^2 + 5x + 4}{x \ln x} \)
Generalized Mean Value Theorem

In order to prove l’Hôpital’s Rule we will need the following.

**Theorem (Generalized Mean Value Theorem)**

Suppose $f$ and $g$ are continuous on $[a, b]$ and differentiable on $(a, b)$ and that $g'(x) \neq 0$ on interval $(a, b)$. Then there exists a number $z \in (a, b)$ such that

$$
\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(z)}{g'(z)}.
$$
Proof of l’Hôpital’s Rule

Proof.

Suppose \( \lim_{x \to c} \frac{f(x)}{g(x)} \) is indeterminate of the form 0/0 and that

\[
\lim_{x \to c} \frac{f'(x)}{g'(x)} = L.
\]

Define the following two functions.

\[
F(x) = \begin{cases} 
  f(x) & \text{if } x \neq c \\ 
  0 & \text{if } x = c 
\end{cases}
\quad \text{and} \quad
G(x) = \begin{cases} 
  g(x) & \text{if } x \neq c \\ 
  0 & \text{if } x = c 
\end{cases}
\]
Indeterminate form $0 \cdot \infty$

**Definition**
A limit of the form

$$\lim_{x \to c} f(x)g(x)$$

where $\lim_{x \to c} f(x) = 0$ and $\lim_{x \to c} g(x) = \pm \infty$ is said to be indeterminate of the form $0 \cdot \infty$.

**Note:**

$$f(x)g(x) = \frac{f(x)}{1} = \frac{g(x)}{1}$$

\[
\begin{align*}
\underbrace{0 \cdot \infty}_{0/0} &= \underbrace{0/0}_{\text{infinite}} \\
\end{align*}
\]
Example

Evaluate the following limit.

\[
\lim_{x \to 0^+} \tan x \ln(\sin x)
\]
Indeterminate form $0^0$

**Definition**
A limit of the form

$$\lim_{x \to c} f(x)^{g(x)}$$

where \(\lim_{x \to c} f(x) = 0\) and \(\lim_{x \to c} g(x) = 0\) is said to be indeterminate of the form $0^0$.

**Note:** If \(y = f(x)^{g(x)}\) then \(\ln y = g(x) \ln(f(x))\) and \(\lim_{x \to c} g(x) \ln(f(x))\) is indeterminate of the form $0 \cdot \infty$.

If \(\lim_{x \to c} g(x) \ln(f(x)) = L\) then \(\lim_{x \to c} f(x)^{g(x)} = e^L\).
Example

Evaluate the following limit.

\[
\lim_{x \to 1^-} (1 - x)^{\ln x}
\]
Indeterminate form $1^\infty$

**Definition**

A limit of the form

$$\lim_{x \to c} f(x)^{g(x)}$$

where $\lim_{x \to c} f(x) = 1$ and $\lim_{x \to c} g(x) = \pm\infty$ is said to be **indeterminate** of the form $1^\infty$.

**Note:** If $y = f(x)^{g(x)}$ then $\ln y = g(x) \ln(f(x))$ and $\lim_{x \to c} g(x) \ln(f(x))$ is indeterminate of the form $0 \cdot \infty$.

If $\lim_{x \to c} g(x) \ln(f(x)) = L$ then $\lim_{x \to c} f(x)^{g(x)} = e^L$. 
Example

Evaluate the following limit.

$$\lim_{{x \to 0^+}} (1 + 3x)^{\cot x}$$
Indeterminate form $\infty^0$

**Definition**
A limit of the form

$$\lim_{x \to c} f(x)^{g(x)}$$

where $\lim_{x \to c} f(x) = \pm \infty$ and $\lim_{x \to c} g(x) = 0$ is said to be indeterminate of the form $\infty^0$.

**Note:** If $y = f(x)^{g(x)}$ then $\ln y = g(x) \ln(f(x))$ and $\lim_{x \to c} g(x) \ln(f(x))$ is indeterminate of the form $0 \cdot \infty$.

If $\lim_{x \to c} g(x) \ln(f(x)) = L$ then $\lim_{x \to c} f(x)^{g(x)} = e^L$. 
Example

Evaluate the following limit.

\[ \lim_{x \to \infty} (1 + e^x)^{1/x} \]
**Indeterminate form** $\infty - \infty$

**Definition**
A limit of the form

$$\lim_{x \to c} (f(x) - g(x))$$

where $\lim_{x \to c} f(x) = \infty$ and $\lim_{x \to c} g(x) = \infty$ is said to be **indeterminate of the form** $\infty - \infty$.

**Example**
Evaluate the following limit.

$$\lim_{x \to \infty} (\sinh x - \cosh x)$$
Homework

Read Section 3.2 and work exercises 1–51 odd.