Millersville University  
Mathematics Department  
MATH 162, Calculus II, Exam  
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Please answer the following questions. Answers without justifying work will receive no credit. Answers must be expressed in exact, simplified form unless a decimal approximation is asked for. Partial credit will be given as appropriate, do not leave any problem blank. The point values of the problems are stated within parentheses.

1. (6 points) Find the area of the region in the second quadrant of the plane bounded by the cardioid \( r = 3(1 - \cos \theta) \).

\[
\begin{align*}
A &= \frac{1}{2} \int_{\pi/2}^{\pi} r^2 \, d\theta - \frac{1}{2} \int_{\pi/2}^{\pi} \left[3(1 - \cos \theta)\right]^2 \, d\theta \\
&= \frac{3}{2} \int_{\pi/2}^{\pi} \left(1 - 2\cos \theta + \cos^2 \theta\right) \, d\theta \\
&= \frac{3}{2} \int_{\pi/2}^{\pi} \left[\frac{3}{2} - 2\cos \theta + \frac{1}{2} \cos 2\theta\right] \, d\theta \\
&= \left[\frac{3}{2} \theta - 2\sin \theta + \frac{1}{4} \sin 2\theta\right]_{\pi/2}^{\pi} \\
&= \frac{3}{2} \left(\frac{\pi}{2} - \pi + \frac{\pi}{2}\right) \\
&= \frac{3}{2} \left(\frac{\pi}{2} + \pi\right) \\
A &\approx 17.063
\end{align*}
\]

2. (5 points each) Differentiate the following expressions with respect to \( x \). Simplify your results.

(a) \( y = x \sin^{-1} x - \sqrt{1 - x^2} \)

\[
\begin{align*}
\frac{dy}{dx} &= \frac{d}{dx} \left[x \sin^{-1} x\right] - \frac{d}{dx} \left[\sqrt{1 - x^2}\right] \\
&= \sin^{-1} x + \frac{x}{\sqrt{1 - x^2}} - \frac{-x}{\sqrt{1 - x^2}} \\
&= \sin^{-1} x + \frac{2x}{\sqrt{1 - x^2}}
\end{align*}
\]
(b) \( y = e^{\sin x} \)

\[
\frac{dy}{dx} = \left( \cos^2 x \right) e^{\sin x}.
\]

3. (5 points) Find the slope of the tangent line to the curve given in polar coordinates by the equation \( \rho = 3 - \cos \theta \) when \( \theta = 5\pi/6 \).

\[
\frac{dy}{d\theta} = \frac{\frac{d\rho}{d\theta} \sin \theta + \rho \cos \theta}{\frac{d\rho}{d\theta} \cos \theta - \rho \sin \theta}.
\]

\[
\frac{d\rho}{d\theta} = (\sin \theta)(\sin \theta) + (3 - \rho \cos \theta) \cos \theta
\]

\[
(\sin \theta)(\cos \theta) - (3 - \rho \cos \theta) \sin \theta.
\]

\[
\frac{\sin^2 \theta - \cos^2 \theta + 3 \cos \theta}{2 \sin^2 \theta - \rho \sin \theta}.
\]

\[
\theta = \frac{5\pi}{6},
\]

\[
\frac{dy}{d\theta} = \frac{\left( \frac{1}{2} \right)^2 - \left( \frac{3}{2} \right)^2 + 3 \left( \frac{\sqrt{3}}{2} \right)^2}{\left( \frac{1}{2} \right) \left( \frac{\sqrt{3}}{2} \right) - \left( \frac{3}{2} \right) \left( \frac{\sqrt{3}}{2} \right)}, \quad \frac{1}{\rho^2} = \frac{1 + 3\sqrt{3}}{3 + \sqrt{3}}.
\]

\[
\approx 1.3014.
\]
4. (6 points each) Evaluate the following definite and indefinite integrals. Simplify your results.

(a) \[ \int_{0}^{\infty} \frac{e^{2x}}{1 + e^{4x}} \, dx \]

Let \( u = e^{2x} \)
\[ \frac{1}{2} \, du = e^{2x} \, dx \]

\[ \lim_{M \to \infty} \int_{0}^{M} \frac{e^{2x}}{1 + e^{4x}} \, dx \]
\[ = \lim_{M \to \infty} \frac{1}{2} \left( \frac{e^{2M}}{1 + e^{4M}} \right) \]
\[ = \frac{1}{2} \lim_{M \to \infty} \left( \tan^{-1} e^{2M} - \tan^{-1} 1 \right) \]
\[ = \left( \frac{\pi}{2} - \frac{\pi}{4} \right) \frac{1}{2} \]
\[ = \frac{\pi}{8} \]

(b) \[ \int \sin^3(mx) \, dx, \ m \text{ is a constant} \]

\[ = \int \left( 1 - \cos^2(mx) \right) \sin(mx) \, dx \]

Let \( u = \cos(mx) \)
\[ \frac{1}{m} \, du = \sin(mx) \, dx \]

\[ = -\frac{1}{m} \int 1 - u^2 \, du \]
\[ = -\frac{1}{m} \left( u - \frac{1}{3} u^3 \right) + C \]
\[ = -\frac{1}{3m} \cos^3(mx) - \frac{1}{m} \cos(mx) + C \]
5. (5 points each) Determine whether the series is absolutely convergent, conditionally convergent, or divergent. You do not need to find the values of the convergent series. You must justify your answers.

(a) \[ \sum_{n=1}^{\infty} \frac{\sqrt{n^2 - 1}}{n^3 + 2n^2 + 5} \]

Let \( L = \frac{1}{n} \infty \).

\[
\lim_{n \to \infty} \frac{\sqrt{n^2 - 1}}{n^3 + 2n^2 + 5} = \lim_{n \to \infty} \frac{\sqrt{n^2 - 1}}{n^3 + 2n^2 + 5} = 1, \quad \text{by the limit comparison test with the convergent } \frac{1}{n^3} \text{ series,}
\]
the given series converges absolutely.

(b) \[ \sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2} \]

Let \( f(x) = e^{1/x} \).

\[
\int_{1}^{\infty} f(x) \, dx = \lim_{M \to \infty} \left[ \frac{e^{1/u}}{u^2} \right]_{1}^{M} = \lim_{M \to \infty} \left( e^{1/M} - e^{-1} \right) = \lim_{M \to \infty} \left( e^{1/M} - e^{-1} \right) = -(1 - e) \text{ which converges.}
\]

By the integral test, the given series is absolutely convergent.
6. (5 points each) Evaluate the following limits.

(a) \( \lim_{x \to \pi} \frac{\sin x}{x^2 - \pi^2} \)

\[
\begin{align*}
&= \lim_{x \to \pi} \frac{\sin x}{2x} \\
&= \frac{\sin \pi}{2\pi} \\
&= \frac{0}{2\pi} \\
&= 0
\end{align*}
\]

(b) \( \lim_{x \to 0^+} \left( \frac{1}{x} - \frac{1}{e^x - 1} \right) \)

\[
\begin{align*}
&= \lim_{x \to 0^+} \left( \frac{e^x - 1}{x(e^x - 1)} \right) \\
&= \lim_{x \to 0^+} \frac{e^x - 1}{e^x - 1 + x e^x} \\
&= \lim_{x \to 0^+} \frac{e^x}{e^x + x e^x} \\
&= \frac{1}{2}
\end{align*}
\]
7. (6 points) Find the volume of the solid of revolution generated if the region bounded
by
\[ y = \sqrt{x^2 + 1}, \quad x = 1, \quad x = 3, \quad \text{and} \quad y = 0 \]
is revolved about the y-axis.
\[
V = \pi \int_{1}^{3} y \sqrt{y^2 + 1} \, dx
\]
\[
= \pi \int_{1}^{3} 2\sqrt{y^2 + 1} \, dx
\]
\[
= \pi \left[ \frac{2}{3} (y^2 + 1)^{3/2} \right]_{1}^{3}
\]
\[
= \frac{4\pi}{3} \left( 22 \sqrt{22} - \sqrt{2} \right) \approx 603.087
\]

8. (6 points) Consider the equation
\[ y = (x + 1)^{3/2}. \]

Find the arc length of the portion of the graph of this curve between \( x = 0 \) and \( x = 1 \).
\[
L = \int_{0}^{1} \sqrt{1 + \left( \frac{3}{2} (x + 1)^{1/2} \right)^2} \, dx
\]
\[
= \int_{0}^{1} \sqrt{1 + \frac{9}{4} (x + 1)} \, dx
\]
\[
= \int_{0}^{1} \sqrt{\frac{4x + 5}{4}} \, dx
\]
\[
= \frac{1}{2^2} \int_{0}^{1} \sqrt{4x + 5} \, dx
\]
\[
= \frac{1}{2^2} \left[ \frac{2}{3} \cdot \frac{4}{7} (4x + 5)^{3/2} \right]_{0}^{1}
\]
\[
= \frac{1}{2^2} \left( 22 \sqrt{22} - 13 \sqrt{13} \right) \approx 2.0859
\]
9. (5 points) A spring is naturally 10 in. long, and a force of 5 lb is required to compress it to a length of 6 in. How much work is done stretching it from its natural length to a length of 13 in.?

\[ W = \int_0^5 \frac{5}{4} x \, dx = \frac{5}{8} x^2 \bigg|_0^5 \]

\[ = \frac{5}{8} (5)^2 \]

\[ W = \frac{4 \times 5^2}{8} \times 5 = 5.625 \text{ ft lb} \]

10. (5 points each) Use known Taylor series to find Taylor series for the following functions.

(a) \( f(x) = x^2 e^{-x/2} \)

\[ = x^2 \sum_{n=0}^{\infty} \frac{(-x/2)^n}{n!} \]

\[ = \sum_{n=0}^{\infty} \frac{(-1)^n}{n} x^{n+2} \]

(b) \( g(x) = \cos(x^4) \)

\[ = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{4n} \]

\[ = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{8n} \]
11. (5 points each) Consider the function, \( f(x) = \ln x \).

(a) Find the Taylor polynomial of order 3 centered at \( c = 1 \) for \( f(x) \).

\[
\begin{align*}
T_2(x) &= x - \frac{1}{1!}(x-1) + \frac{1}{2!}(x-1)^2 + \frac{1}{3!}(x-1)^3 \\
&= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3
\end{align*}
\]

(b) What is the maximum error in using the Taylor polynomial of order 3 centered at \( c = 1 \) to estimate \( \ln 1.1 \)?

\[
|R_2(1.1)| \leq \left| \frac{-e^{1/2}}{2!} (1.1-1)^2 \right| = \frac{1}{2} (0.1)^2 = 0.00025
\]
12. (5 points each) Consider the power series \( \sum_{n=0}^{\infty} \left( \frac{x-3}{4} \right)^n \).

(a) Find the radius of convergence of the power series.

\[
\lim_{n \to \infty} \left| \frac{\left( \frac{x-3}{4} \right)^{n+1}}{\left( \frac{x-3}{4} \right)^n} \right| = \lim_{n \to \infty} \left| \frac{x-3}{4} \right| = \left| \frac{x-3}{4} \right| < 1
\]

\[
|x-3| < 4 \quad \Rightarrow \quad -1 < x < 7
\]

(b) Find the interval of convergence of the power series.

\[
-4 < x - 3 < 4
\]

\[-1 < x < 7
\]

Left: \( x = -1 \), \( \sum_{n=0}^{\infty} \left( \frac{-1-3}{4} \right)^n = \sum_{n=0}^{\infty} (-1)^n \) diverges by the

Right: \( x = 7 \), \( \sum_{n=0}^{\infty} \left( \frac{7-3}{4} \right)^n = \sum_{n=0}^{\infty} 1 \) diverges by the

\[
\text{interval of convergence is } -1 < x < 7
\]