Millersville University  
Mathematics Department  
MATH 162, Calculus II, Test 4  
April 27, 2001

Please answer the following questions. Answers without justifying work will receive no credit. Answers must be expressed in exact, simplified form unless a decimal approximation is asked for. Partial credit will be given as appropriate, do not leave any problem blank. The point values of the problems are stated within parentheses.

1. (6 points each) Use known Taylor series to find Taylor series for the following functions.

(a) \( e^{2x/3} \)

\[
e^{\frac{2x}{3}} = \sum_{n=0}^{\infty} \frac{\left(\frac{2x}{3}\right)^n}{n!}
\]

\[
= \sum_{n=0}^{\infty} \frac{2^n}{3^n n!} x^n
\]

(b) \( 2x \sin(\pi x) \)

\[
2x \sin(\pi x) = 2x \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (\pi x)^{2n+1}
\]

\[
= \sum_{n=0}^{\infty} \frac{2(-1)^n}{(2n+1)!} \pi^{2n+1} x^{2n+1}
\]

\[
= \sum_{n=0}^{\infty} \frac{2(-1)^n \pi^{2n+1}}{(2n+1)!} x^{2n+2}
\]

(c) \( \ln(1-x), |x| < 1 \)

\[
\ln(1-x) = \int \frac{-1}{1-x} \, dx = \int \sum_{n=0}^{\infty} (-1)^n x^n \, dx
\]

\[
= \sum_{n=0}^{\infty} \frac{-1}{n+1} x^{n+1}
\]
(d) \( \int_0^\infty e^{-t^2} \, dt = \left( \sum_{n=0}^{\infty} \frac{1}{n!} \right) \int_0^\infty (-t^2)^n \, dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! (2n+1)} x^{2n+1} \)

2. (8 points) A projectile is fired with a velocity of 500 m/s from a gun 1.5 m above the ground at an angle of 30° above horizontal. How far from the gun will the projectile land?

\( x = (500 \cos 30^\circ) t \)

\( y = \frac{x^2}{2} + (500 \sin 30^\circ) t - \frac{1}{2} (9.81) t^2 \)

When the projectile hits the ground, \( y = 0 \).

\( \frac{3}{2} + 250 t - \frac{1}{2} (9.81) t^2 = 0 \)

\( t = \frac{250 \pm \sqrt{(250)^2 + 4(\frac{1}{2})(9.81)(\frac{3}{2})}}{9.81} \)

\( t = \frac{250 + \sqrt{(250)^2 + 3(9.81)}}{9.81} \approx 50.974 \)

\( x = 250 \sqrt{3} t \approx 22072.56 \text{ m} \)
3. (6 points each) Consider the function \( f(x) = \frac{1}{x} \).

(a) Find the Taylor polynomial of order 3 centered at \( c = 1 \) for \( f(x) \).
\[
\begin{align*}
\phi_0(x) &= \frac{1}{x} \quad \phi_0(1) = 1 \\
\phi_0'(x) &= -\frac{1}{x^2} \quad \phi_0'(1) = -1 \\
\phi_0''(x) &= \frac{2}{x^3} \quad \phi_0''(1) = 2 \\
\phi_0'''(x) &= -\frac{6}{x^4} \quad \phi_0'''(1) = -6 \\
\mathcal{P}_3(x) &= 1 - (x - 1) + (x - 1)^2 - (x - 1)^3
\end{align*}
\]

(b) What is the maximum error in using a Taylor polynomial of order 3 centered at \( c = 1 \) to estimate \( 1/1.2 \)?
\[
\mathcal{R}_3(x) = \frac{2^4 / 4!}{x^5} (x - 1)^4 \\
|\mathcal{R}_3(1.2)| \leq (1.2 - 1)^4 = (0.2)^4 = 0.0016
\]

(c) Find the Taylor series centered at \( c = 1 \) for \( f(x) \).
\[
\begin{align*}
\phi_0(x) &= \frac{1}{x} = \frac{1}{1-(1-x)} = \sum_{n=0}^{\infty} (1-x)^n \\
&= \sum_{n=0}^{\infty} (-1)^n (x-1)^n
\end{align*}
\]
4. (6 points each) An object is traveling along the path described parametrically by

\[ x = \cos^3 t \]
\[ y = \sin^3 t \]

for \(0 \leq t \leq \pi\).

(a) Graph the path of the object and label its orientation.

(b) What is the object's speed when \(t = 1\)?

\[ \frac{dx}{dt} = -3\cos^2 t \sin t \]
\[ \frac{dy}{dx} = 3\sin^2 t \cos t \]

\[ \Delta = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dx}\right)^2} \]
\[ \Delta = \sqrt{9\cos^4 t \sin^2 t + 9\sin^4 \cos^2 t} \]
\[ = 3|\cos t| \sin t \sqrt{\cos^2 t + \sin^2 t} \]
\[ = 3|\cos t| \sin t \]

\[ \Delta = 3(0.911)(0.911) \approx 1.3639 \]
(c) Find the distance that the object travels.

\[
L = \int_0^{\pi/2} 2 \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} \, dt
= 2 \int_0^{\pi/2} \sqrt{\frac{3}{2} \cos^2 t + \frac{3}{2} \sin^2 t} \, dt
= 2 \int_0^{\pi/2} \sqrt{\frac{3}{2} t} \, dt
= 2 \left[ \frac{2^{3/2} t^{1/2}}{2} \right]_0^{\pi/2}
= 3
\]

5. (6 points each) Consider the power series \( \sum_{k=0}^{\infty} \frac{-3}{\sqrt{k}} \left( \frac{x}{2} \right)^k \).

(a) Find the radius of convergence of the series.

Using the Ratio Test: 
\[
\lim_{k \to \infty} \left| \frac{\frac{-3}{\sqrt{k+1}} \left( \frac{x}{2} \right)^{k+1}}{\frac{-3}{\sqrt{k}} \left( \frac{x}{2} \right)^k} \right| = \left| \frac{x}{2} \right| < 1
\]

\(|x| < 2 = R\), the ratio test shows convergence.

(b) Find the interval of convergence of the series.

Let \( x = -2 \), \( \sum_{k=0}^{\infty} \frac{-3}{\sqrt{k}} (-1)^k \) converge by AST.

Let \( x = 2 \), \( \sum_{k=0}^{\infty} \frac{-3}{\sqrt{k}} \) diverge.

Interval of convergence: \(-2 \leq x < 2\).
6. (5 points each) Determine of the following infinite series converge absolutely, converge conditionally, or diverge. You must justify your answers using a convergence or divergence test.

(a) $\sum_{k=1}^{\infty} \frac{(-1)^k k}{k^3 + 1}$

$$0 \leq \left| \frac{(-1)^{k+1} k}{k^3 + 1} \right| \leq \frac{k}{k^3 + 1} \leq \frac{1}{k^2}$$

Converge absolutely by the comparison test.

(b) $\sum_{k=1}^{\infty} \frac{\sin k}{k^{3/2}}$

$$0 \leq \left| \frac{\sin k}{k^{3/2}} \right| \leq \frac{1}{k^{3/2}}$$

Converge absolutely by the comparison test.

(c) $\sum_{k=2}^{\infty} \frac{7}{k \ln k}$

$$\int_{2}^{M} \frac{1}{k \ln k} \, dk = \lim_{M \to \infty} \int_{2}^{M} \frac{1}{k \ln k} \, dk$$

$$= \lim_{M \to \infty} \ln(\ln k) \bigg|_{2}^{M}$$

$$= \lim_{M \to \infty} (\ln(\ln M) - \ln(\ln 2))$$

Diverges.
(d) \[ \sum_{k=1}^{\infty} \frac{k^2 + 4}{k^3 + 2k + 3} \]

Let \[ y_n = \frac{1}{k} \]

\[ \lim_{k \to \infty} \frac{k^2 + 4}{k^3 + 2k + 3} \cdot \frac{1/k}{1/k} = \lim_{k \to \infty} \frac{2k^3 + 4k}{k^3 + 2k^2 + 3} = 1 \]

Diverges by the limit comparison test.