Prove the following statements. Your answer will be revised to check correctness, completeness, and use of mathematical concepts in your answer. Please show all work and write on your work neatly; answers without supporting work will receive no credit. The precise amount of the problem set is given in the problem.

1. (10 points) Fill in the blanks whether the following statements are true or false. Your arguments should be concise.
   \[
   \text{True or False: } \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges.}
   \]
   \[
   \text{True or False: } \sum_{n=1}^{\infty} \frac{1}{2^n} \text{ converges.}
   \]
   \[
   \text{True or False: } \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \text{ converges.}
   \]
   \[
   \text{True or False: } \sum_{n=1}^{\infty} \frac{1}{n^{1/2}} \text{ diverges.}
   \]
   \[
   \text{True or False: } \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges.}
   \]
   \[
   \text{True or False: } \sum_{n=1}^{\infty} \frac{1}{n^{1/3}} \text{ converges.}
   \]
   \[
   \text{True or False: } \sum_{n=1}^{\infty} \frac{1}{n^{2/3}} \text{ diverges.}
   \]

2. (10 points) Find the interval and interval of convergence of the following power series.
   \[
   \sum_{n=0}^{\infty} \frac{x^n}{n!}
   \]
   \[
   \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}
   \]
   \[
   \sum_{n=0}^{\infty} \frac{x^n}{n^n}
   \]

3. (10 points) Let the function defined by
   \[
   f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}
   \]

   a. Find the Taylor series expansion of \( f(x) \) about \( x = 0 \).
   b. Find the radius of convergence of the Taylor series.
   c. Compute \( f(1) \).
   d. Compute \( f''(0) \).
   e. Compute \( f^{(n)}(0) \) for \( n \geq 1 \).

4. (10 points) Let the function defined by
   \[
   g(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}
   \]

   a. Find the Taylor series expansion of \( g(x) \) about \( x = 0 \).
   b. Find the radius of convergence of the Taylor series.
   c. Compute \( g(1) \).
   d. Compute \( g''(0) \).
   e. Compute \( g^{(n)}(0) \) for \( n \geq 1 \).

5. (10 points) Let the function defined by
   \[
   h(x) = \sum_{n=0}^{\infty} \frac{x^n}{n^n}
   \]

   a. Find the Taylor series expansion of \( h(x) \) about \( x = 0 \).
   b. Find the radius of convergence of the Taylor series.
   c. Compute \( h(1) \).
   d. Compute \( h''(0) \).
   e. Compute \( h^{(n)}(0) \) for \( n \geq 1 \).
6. (3 points) Find the exact value of the root of the following roots.

$$\sqrt[3]{-27}$$

$$\sqrt[4]{16}$$

$$\sqrt[5]{32}$$
9. (1 point each) Determine the infinite series.

\[ \sum_{n=1}^{\infty} \frac{1}{n^2} \]

(a) Determine the minimum number of terms which must be added to form a partial sum which is accurate to five decimal places.

\[ \sum_{n=1}^{k} \frac{1}{n^2} \approx 1.64493 \]

(b) Approximate the sum of the series to five accurate decimal places.

\[ S = \frac{1}{2} \sum_{n=1}^{k} \frac{1}{n^2} = 1.64493 \]
8. (a) Find the Taylor polynomial \( P_N(x) \) of order \( N = 1 \) for the function \( f(x) \) at a = 0.

\[
P_N(x) = f(0) + \frac{f'(0)}{1!} x = 0 + \frac{0}{1!} x = 0
\]

(b) Use the Taylor remainder \( R_N(x) \) for \( N = 1 \) to estimate the error in the approximation.

\[
R_1(x) = \frac{f''(c)}{2!} x^2 = \frac{0}{2!} x^2 = 0
\]

\[
|R_1(x)| < \frac{1}{2} x^2
\]

\[
\therefore \quad |R_1(x)| < \frac{1}{2} 0^2 = 0
\]