1. The orbital period of the planet Mars is approximately 686.98 days. The mass of the sun is approximately $1.99 \times 10^{30}$ kg. Find the length of the major axis of Mars’ orbit in units of meters and miles.

Note that

\[ T: \text{orbital period of Mars,} \]
\[ 686.98 \text{ days} = (686.98)(86400) \text{ sec} = 5.9355072 \times 10^7 \text{ sec.} \]

\[ G: \text{universal gravitational constant,} \]
\[ G = 6.672 \times 10^{-11} \text{ N m}^2/\text{kg}^2 \]

Using Kepler’s third law, \( T^2 = ka^3 \), we have

\[ (5.9355072 \times 10^7)^2 = \frac{4\pi^2}{(6.672 \times 10^{-11})(1.99 \times 10^{30})} a^3 \]
\[ 3.523024572125184 \times 10^{15} = 2.973381415798826 \times 10^{-19} a^3 \]
\[ 1.184854574460536 \times 10^{34} = a^3 \]
\[ a = 2.279755908041359 \times 10^{11} \text{ meters} \]

Thus the major axis of the orbit is

\[ 2a = 4.559511816082718 \times 10^{11} \text{ meters} \]
\[ = 2.833143626880944 \times 10^8 \text{ miles.} \]

2. A satellite in geosynchronous orbit remains in a fixed location above the Earth’s equator. The mass of the Earth is approximately $5.98 \times 10^{24}$ kg. Find the altitude in meters and in miles of a geosynchronous orbit. The length of the sidereal day (the amount of time it takes the Earth to turn once on its axis) is 23.93 hours.

We want the orbital period of the satellite to be a sidereal day. Note that

\[ T: \text{orbital period of satellite,} \]
\[ 23.93 \text{ hours} = (23.93)(3600) \text{ sec} = 86148 \text{ sec.} \]
\(G\): universal gravitational constant,

\[ G = 6.672 \times 10^{-11} \text{ N m}^2/\text{kg}^2 \]

Using Kepler’s third law, \(T^2 = ka^3\), we have

\[ \frac{86148^2}{(5.98 \times 10^{24})a^3} = \frac{4\pi^2}{(6.672 \times 10^{-11})(5.98 \times 10^{24})a^3} \]

\[ 7.421477904 \times 10^9 = 9.894697353578034 \times 10^{-14}a^3 \]

\[ 7.500459729893922 \times 10^{22} = a^3 \]

\[ a = 4.21725 \times 10^7 \text{ meters} \]

\[ = 26204.7 \text{ miles} \]

The mean radius of the Earth is \(R = 6370949 \text{ meters} \) (3958.72 miles). Thus the altitude of the satellite will be \(3.58016 \times 10^7 \text{ meters} \) (22246 miles).

3. Let \(r(t)\) be the position of an object as a function of time. Let the mass of the object be described by the function \(m(t)\). The vector-valued function \(M(t) = m(t)r'(t)\) is the linear momentum of the object. The vector-valued function \(F(t) = (m(t)r'(t))'\) is called the force vector. The angular momentum about the origin for the object is \(L(t) = r(t) \times M(t)\). The torque about the origin is \(\tau(t) = r(t) \times F(t)\).

(a) Show that if the force vector is always zero, then the linear momentum is constant (this is the law of conservation of linear momentum).

Suppose \(0 = F(t) = \frac{d}{dt}M(t)\) then \(M(t) = c\) a constant vector.

(b) Show that the torque is the derivative of angular momentum.

\[
\frac{d}{dt}L(t) = \frac{d}{dt}(r(t) \times M(t))
\]

\[
= (r'(t) \times M(t)) + (r(t) \times M'(t))
\]

\[
= (r'(t) \times m(t)r'(t)) + (r(t) \times F(t))
\]

\[
= m(t)(r'(t) \times r'(t)) + (r(t) \times F(t))
\]

\[
= 0 + \tau(t)
\]

\[
= \tau(t)
\]

(c) Show that if the torque is always zero, then the angular momentum is constant (this is the law of conservation of angular momentum).

If \(\tau(t) = 0\) then \(\frac{d}{dt}L(t) = 0\) and \(L(t) = c\) a constant vector.