Please answer the following questions. Partial credit will be given as appropriate, do not leave any problem blank. Each problem is worth ten points. Your completed assignment will be due at class time on Tuesday, September 27, 2005.

1. Consider the vectors

\[ u(t) = t^2i + 6tj + tk \]
\[ v(t) = ti - 5tj + 4t^2k \]

(a) Find the values of \( t \) for which \( u(t) \) and \( v'(t) \) are orthogonal.

\[ 0 = u(t) \cdot v'(t) = \langle t^2, 6t, t \rangle \cdot \langle 1, -5, 8t \rangle = t^2 - 30t + 8t^2 = 9t^2 - 30t \]

Hence \( t = 0 \) or \( t = 10/3 \).

(b) Find \( (u(t) \times v(t))' \).

\[
(u(t) \times v(t))' = (u'(t) \times v(t)) + (u(t) \times v'(t)) \\
= \begin{vmatrix}
i & j & k \\
t & 6t & 1 \\
2t & 6 & t \\
\end{vmatrix} + \begin{vmatrix}
i & j & k \\
t^2 & 6t & t \\
t^2 & 6t & 1 \\
\end{vmatrix} \\
= \langle 5t + 24t^2, t - 8t^3, -6t - 10t^2 \rangle + \langle 5t + 48t^2, t - 8t^3, -6t - 5t^2 \rangle \\
= \langle 10t + 72t^2, 2t - 16t^3, -12t - 15t^2 \rangle
\]

2. A particle \( P \) is moving with an acceleration \( 12tj + 5k \) at time \( t \). If at \( t = 1 \), the particle is at coordinates \( (-1, 3, 3/2) \) and its velocity is \( 3i - 2j + 4k \), find its position vector \( r(t) \).

\[
v(t) = \int a(t) \, dt \\
= \int \langle 0, 12t, 5 \rangle \, dt \\
= \langle 0, 6t^2, 5t \rangle + c \\
v(1) = \langle 0, 6, 5 \rangle + c \\
f1 = \langle 3, -2, 4 \rangle = \langle 0, 6, 5 \rangle + c \\
c = \langle 3, -8, -1 \rangle
\]
Hence the velocity vector of the particle is given by \( \mathbf{v}(t) = (3, 6t^2 - 8, 5t - 1) \).

\[
\mathbf{r}(t) = \int \mathbf{v}(t) \, dt \\
= \int (3, 6t^2 - 8, 5t - 1) \, dt \\
= (3t, 2t^3 - 8t, \frac{5}{2}t^2 - t) + \mathbf{d} \\
\mathbf{r}(1) = (3, -6, -\frac{3}{2}) + \mathbf{d} \\
\langle -1, 3, \frac{3}{2} \rangle = (3, -6, -\frac{3}{2}) + \mathbf{d} \\
\mathbf{d} = \langle -4, 9, 0 \rangle 
\]

Thus the position vector of the particle is the function \( \mathbf{r}(t) = (3t, 2t^3 - 8t + 9, \frac{5}{2}t^2 - t) \).

3. Find the curvature of the curve described by

\[
\mathbf{r}(t) = \langle 2t^3, t^4, 4t^2 \rangle. 
\]

\[
\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} \\
= \frac{\|\langle 6t^2, 4t^3, 8t \rangle \times \langle 12t, 12t^2, 8 \rangle\|}{\|\langle 6t^2, 4t^3, 8t \rangle\|^3} \\
= \frac{\|\langle -64t^3, 48t^2, 24t^4 \rangle\|}{(64t^2 + 36t^4 + 16t^6)^{3/2}} \\
= \frac{\|\langle -64t^3, 48t^2, 24t^4 \rangle\|}{8t^3(16 + 9t^2 + 4t^4)^{3/2}} \\
= \frac{\|\langle -8t, 6, 3t^2 \rangle\|}{t(16 + 9t^2 + 4t^4)^{3/2}} \\
= \frac{\sqrt{9t^4 + 64t^2 + 36}}{t(16 + 9t^2 + 4t^4)^{3/2}} 
\]

4. Find the \( x \) coordinates of the points on the graph of \( y = x^3 - 3x \) at which the curvature is a maximum.

\[
\kappa(x) = \frac{|y''|}{(1 + (y')^2)^{3/2}} \\
= \frac{|6x|}{(1 + (3x^2 - 3)^2)^{3/2}} \\
= \frac{|6x|}{(9x^4 - 18x^2 + 10)^{3/2}} 
\]
Note that this is an even function, thus if a maximum occurs at some \( x > 0 \) a maximum is also achieved at \( -x < 0 \).

\[
\kappa'(x) = \frac{-270x^4 + 216x^2 + 60}{(9x^4 - 18x^2 + 10)^{5/2}}
\]
\[
0 = \frac{-270x^4 + 216x^2 + 60}{(9x^4 - 18x^2 + 10)^{5/2}}
\]
\[
= -270x^4 + 216x^2 + 60
\]
\[
x = \sqrt{\frac{1}{15}(6 + \sqrt{86})}
\]
\[
\kappa''(x) = \frac{324x(45x^6 - 63x^4 - 14x^2 + 30)}{(9x^4 - 18x + 10)^{7/2}}
\]
\[
\kappa'' \left( \sqrt{\frac{1}{15}(6 + \sqrt{86})} \right) < 0
\]

Thus the maximum curvature occurs when \( x = \pm \sqrt{\frac{1}{15}(6 + \sqrt{86})} \).