Today we will apply the double integral to the problems of finding
- the **area** of regions in the plane,
- the **volume** of a bounded region in three dimensions,
- the **center of mass** of a region in the plane.
If $R$ is a region in the $xy$-plane, the area $A$ of $R$ is

$$A = \iint_{R} 1 \, dA$$

provided the value of the double integral is finite.
Example

Find the area of the triangular region with vertices at (0, 2), (3, 2), and (1, 1).
Solution

The region $R$ is bounded by the graphs of the lines $x = 2 - y$ and $x = 2y - 1$ for $1 \leq y \leq 2$.

$$A = \iint_R 1 \, dA$$

$$= \int_1^2 \int_{2-y}^{2y-1} 1 \, dx \, dy$$

$$= \int_1^2 (2y - 1) - (2 - y) \, dy$$

$$= \int_1^2 3y - 3 \, dy$$

$$= \frac{3}{2} y^2 - 3y \bigg|_1^2$$

$$= 6 - 6 - \left( \frac{3}{2} - 3 \right) = \frac{3}{2}$$
Example

Find the area of the bounded region in the $xy$-plane between $y = \sqrt{x}$ and $y = x^2$. 
Solution

\[ A = \int \int_{R} 1 \, dA \]

\[ = \int_{0}^{1} \int_{x^2}^{\sqrt{x}} 1 \, dy \, dx \]

\[ = \int_{0}^{1} \sqrt{x} - x^2 \, dx \]

\[ = \left[ \frac{2}{3} x^{3/2} - \frac{1}{3} x^3 \right]_{0}^{1} \]

\[ = \frac{1}{3} \]
Example

Find the area of the bounded region in the $xy$-plane between $y = x$ and $x = y^2 - y$. 
Solution

\[ A = \int\int_R 1 \, dA \]
\[ = \int_0^2 \int_{y^2 - y}^y 1 \, dx \, dy \]
\[ = \int_0^2 y - (y^2 - y) \, dy \]
\[ = \int_0^2 2y - y^2 \, dy \]
\[ = y^2 - \frac{1}{3}y^3 \bigg|_0^2 \]
\[ = 4 - \frac{8}{3} = \frac{4}{3} \]
Volume of a Solid Region

If $f(x, y) \geq 0$ for all $(x, y)$ in a region $R$, the volume of the solid below the surface $z = f(x, y)$ and above the region $R$ in the $xy$-plane is

$$V = \int \int_{R} f(x, y) \, dA.$$
Volume of a Solid Region

If \( f(x, y) \geq 0 \) for all \((x, y)\) in a region \( R \), the volume of the solid below the surface \( z = f(x, y) \) and above the region \( R \) in the \( xy \)-plane is

\[
V = \iint_R f(x, y) \, dA.
\]

If \( f(x, y) \geq g(x, y) \) for all \((x, y)\) in a region \( R \), the volume of the solid below the surface \( z = f(x, y) \) and above the surface \( z = g(x, y) \) within the the region \( R \) in the \( xy \)-plane is

\[
V = \iint_R (f(x, y) - g(x, y)) \, dA.
\]
Example

Find the volume of the solid region below the graph of \( z = e^{y^2} \) and above the triangle in the \( xy \)-plane with vertices at \((0, 0), (1, 1), (0, 1)\).
Solution

\[ V = \iiint_R f(x, y) \, dA \]
\[ = \iiint_R e^{y^2} \, dA \]
\[ = \int_0^1 \int_0^y e^{y^2} \, dx \, dy \]
\[ = \left. xe^{y^2} \right|_0^y \, dy \]
\[ = \int_0^1 ye^{y^2} \, dy \]
\[ = \frac{1}{2} e^{y^2} \bigg|_0^1 \]
\[ = \frac{1}{2} e - \frac{1}{2} \]
Example

Find the volume of the solid region above the graph of \( z = 2x^2 + y^2 \) and below the graph of \( z = 8 - x^2 - 2y^2 \) and within the rectangle \( R = \{(x, y) \mid 0 \leq x \leq 1, \ 0 \leq y \leq 1\} \).
Solution

\[ V = \int \int_R f(x, y) \, dA \]
\[ = \int_0^1 \int_0^1 (8 - x^2 - 2y^2) - (2x^2 + y^2) \, dx \, dy \]
\[ = \int_0^1 \int_0^1 (8 - 3x^2 - 3y^2) \, dx \, dy \]
\[ = \int_0^1 \left[ 8x - x^3 - 3xy^2 \right]_0^1 \, dy \]
\[ = \int_0^1 (8 - 1 - 3y^2) \, dy \]
\[ = \int_0^1 (7 - 3y^2) \, dy \]
\[ = 7y - y^3 \bigg|_0^1 = 6 \]
Moments and Center of Mass

**lamina**: a thin sheet of material in the shape of a region $R \subset \mathbb{R}^2$.

$\rho(x, y)$: the density of the lamina at $(x, y)$.

$m$: mass of the lamina

$$m = \int\int_{R} \rho(x, y) \, dA.$$  

$M_x$: moment with respect to the $x$-axis

$$M_x = \int\int_{R} y \rho(x, y) \, dA.$$  

$M_y$: moment with respect to the $y$-axis

$$M_y = \int\int_{R} x \rho(x, y) \, dA.$$  

$(\bar{x}, \bar{y})$ center of mass

$$ (\bar{x}, \bar{y}) = \left( \frac{M_y}{m}, \frac{M_x}{m} \right) $$
Example

A lamina with density $\rho(x, y) = y + 3$ is bounded by the curves $x = y^2$ and $x = 4$. Find its mass, moments, and center of mass.
Solution

\[ m = \int_{-2}^{2} \int_{y^2}^{4} (y + 3) \, dx \, dy \]
\[ M_x = \int_{-2}^{2} \int_{y^2}^{4} y(y + 3) \, dx \, dy \]
\[ M_y = \int_{-2}^{2} \int_{y^2}^{4} x(y + 3) \, dx \, dy \]
Solution

\[ m = \int_{-2}^{2} \int_{y^2}^{4} (y + 3) \, dx \, dy = \int_{-2}^{2} (12 + 4y - 3y^2 - y^3) \, dy = 32 \]

\[ M_x = \int_{-2}^{2} \int_{y^2}^{4} y(y + 3) \, dx \, dy \]

\[ M_y = \int_{-2}^{2} \int_{y^2}^{4} x(y + 3) \, dx \, dy \]
Solution

\[ m = \int_{-2}^{2} \int_{y^2}^{4} (y + 3) \, dx \, dy = \int_{-2}^{2} (12 + 4y - 3y^2 - y^3) \, dy = 32 \]

\[ M_x = \int_{-2}^{2} \int_{y^2}^{4} y(y + 3) \, dx \, dy \]

\[ = \int_{-2}^{2} (12y + 4y^2 - 3y^3 - y^4) \, dy = \frac{128}{15} \]

\[ M_y = \int_{-2}^{2} \int_{y^2}^{4} x(y + 3) \, dx \, dy \]
Solution

\[ m = \int_{-2}^{2} \int_{y^2}^{4} (y + 3) \, dx \, dy = \int_{-2}^{2} (12 + 4y - 3y^2 - y^3) \, dy = 32 \]

\[ M_x = \int_{-2}^{2} \int_{y^2}^{4} y(y + 3) \, dx \, dy \]

\[ = \int_{-2}^{2} (12y + 4y^2 - 3y^3 - y^4) \, dy = \frac{128}{15} \]

\[ M_y = \int_{-2}^{2} \int_{y^2}^{4} x(y + 3) \, dx \, dy \]

\[ = \int_{-2}^{2} \left( 24 + 8y - \frac{3}{2}y^4 - \frac{1}{2}y^5 \right) \, dy = \frac{384}{5} \]
Solution

\[ m = \int_{-2}^{2} \int_{y^2}^{4} (y + 3) \, dx \, dy = \int_{-2}^{2} (12 + 4y - 3y^2 - y^3) \, dy = 32 \]

\[ M_x = \int_{-2}^{2} \int_{y^2}^{4} y(y + 3) \, dx \, dy \]
\[ = \int_{-2}^{2} (12y + 4y^2 - 3y^3 - y^4) \, dy = \frac{128}{15} \]

\[ M_y = \int_{-2}^{2} \int_{y^2}^{4} x(y + 3) \, dx \, dy \]
\[ = \int_{-2}^{2} \left( 24 + 8y - \frac{3}{2}y^4 - \frac{1}{2}y^5 \right) \, dy = \frac{384}{5} \]

\((\overline{x}, \overline{y}) = \left( \frac{M_y}{m}, \frac{M_x}{m} \right) = \left( \frac{12}{5}, \frac{4}{15} \right)\)
Moment of Inertia

**Inertia**: tendency of an object to resist a change in motion.

**$I_x$**: moment of inertia about the $x$-axis

$$I_x = \int\int_R y^2 \rho(x, y) \, dA.$$  

**$I_y$**: moment of inertia about the $y$-axis

$$I_y = \int\int_R x^2 \rho(x, y) \, dA.$$
Example

A lamina with density $\rho(x, y) = y + 3$ is bounded by the curves $x = y^2$ and $x = 4$. Find its moments of inertia.
Solution

\[ I_x = \int_{-2}^{2} \int_{y^2}^{4} y^2(y + 3) \, dx \, dy \]

\[ I_y = \int_{-2}^{2} \int_{y^2}^{4} x^2(y + 3) \, dx \, dy \]
Solution

\[ I_x = \int_{-2}^{2} \int_{y^2}^{4} y^2(y + 3) \, dx \, dy \]

\[ = \int_{-2}^{2} 12y^2 + 4y^3 - 3y^4 - y^5 \, dy \]

\[ = 4y^3 + y^4 - \frac{3}{5}y^5 - \frac{1}{6}y^6 \bigg|_{-2}^{2} = \frac{128}{5} \]

\[ I_y = \int_{-2}^{2} \int_{y^2}^{4} x^2(y + 3) \, dx \, dy \]
Solution

\[ I_x = \int_{-2}^{2} \int_{y^2}^{4} y^2(y + 3) \, dx \, dy \]

\[ = \int_{-2}^{2} \left( 12y^2 + 4y^3 - 3y^4 - y^5 \right) \, dy \]

\[ = 4y^3 + y^4 - \frac{3}{5}y^5 - \frac{1}{6}y^6 \bigg|_{-2}^{2} = \frac{128}{5} \]

\[ I_y = \int_{-2}^{2} \int_{y^2}^{4} x^2(y + 3) \, dx \, dy \]

\[ = \int_{-2}^{2} \left( 64 + \frac{64}{3}y - y^6 - \frac{1}{3}y^7 \right) \, dy \]

\[ = 64y + \frac{32}{3}y^2 - \frac{1}{7}y^7 - \frac{1}{24}y^8 \bigg|_{-2}^{2} = \frac{1536}{7} \]
Homework

- Read Section 13.2.
- Exercises: 1–21 odd, 29, 31, 41