The Calculus of Vector-Valued Functions
MATH 311, *Calculus III*

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Limits of Vector-Valued Functions

Definition
For a vector-valued function \( \mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle \), the limit of \( \mathbf{r}(t) \) as \( t \) approaches \( a \) is

\[
\lim_{t \to a} \mathbf{r}(t) = \lim_{t \to a} \langle f(t), g(t), h(t) \rangle = \langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \rangle
\]

provided all of the indicated limits exist. If any of the limits on the right-hand side do not exist, then \( \lim_{t \to a} \mathbf{r}(t) \) does not exist.
Example (1 of 2)

Evaluate the following limit, if it exists.

\[
\lim_{t \to 0} \langle 1 + t^3, te^{-t}, \frac{\sin t}{t} \rangle
\]
Example (1 of 2)

Evaluate the following limit, if it exists.

\[
\lim_{t \to 0} \langle 1 + t^3, te^{-t}, \frac{\sin t}{t} \rangle
\]

\[
= \langle \lim_{t \to 0} 1 + t^3, \lim_{t \to 0} te^{-t}, \lim_{t \to 0} \frac{\sin t}{t} \rangle
\]

\[
= \langle 1, 0, 1 \rangle
\]
Example (2 of 2)

Evaluate the following limit, if it exists.

$$\lim_{t \to 1} \langle \sqrt{t - 1}, t^2 + 1, \frac{t + 1}{t - 1} \rangle$$

Does not exist since $$\lim_{t \to 1} \sqrt{t - 1}$$ does not exist, and $$\lim_{t \to 1} \frac{t + 1}{t - 1}$$ does not exist.
Example (2 of 2)

Evaluate the following limit, if it exists.

\[
\lim_{t \to 1} \langle \sqrt{t - 1}, t^2 + 1, \frac{t + 1}{t - 1} \rangle
\]

\[
\lim_{t \to 1} \langle \sqrt{t - 1}, t^2 + 1, \frac{t + 1}{t - 1} \rangle
= \left\langle \lim_{t \to 1} \sqrt{t - 1}, \lim_{t \to 1} (t^2 + 1), \lim_{t \to 1} \frac{t + 1}{t - 1} \right\rangle
\]

Does not exist since \(\lim_{t \to 1} \sqrt{t - 1}\) does not exist, and \(\lim_{t \to 1} \frac{t + 1}{t - 1}\) does not exist.
Example (2 of 2)

Evaluate the following limit, if it exists.

\[
\lim_{t \to 1} \langle \sqrt{t - 1}, t^2 + 1, \frac{t + 1}{t - 1} \rangle
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\[
\lim_{t \to 1} \langle \sqrt{t - 1}, t^2 + 1, \frac{t + 1}{t - 1} \rangle = \left\langle \lim_{t \to 1} \sqrt{t - 1}, \lim_{t \to 1} (t^2 + 1), \lim_{t \to 1} \frac{t + 1}{t - 1} \right\rangle
\]

Does not exist since

\[
\lim_{t \to 1} \sqrt{t - 1} \quad \text{does not exist, and}
\]

\[
\lim_{t \to 1} \frac{t + 1}{t - 1} \quad \text{does not exist.}
\]
Continuity (1 of 2)

Definition

The vector-valued function \( r(t) = \langle f(t), g(t), h(t) \rangle \) is **continuous** at \( t = a \) whenever

\[
\lim_{t \to a} r(t) = r(a).
\]
Continuity (1 of 2)

Definition
The vector-valued function \( \mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle \) is \textbf{continuous} at \( t = a \) whenever

\[ \lim_{{t \to a}} \mathbf{r}(t) = \mathbf{r}(a). \]

The limit must exist at \( t = a \) and equal the value of the function at \( t = a \).
Continuity (2 of 2)

Theorem
A vector-valued function $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ is continuous at $t = a$ if and only if all of $f$, $g$, and $h$ are continuous at $t = a$.
Theorem

A vector-valued function \( \mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle \) is continuous at \( t = a \) if and only if all of \( f, g, \) and \( h \) are continuous at \( t = a \).

Example

1. Determine the values of \( t \) for which \( \mathbf{r}(t) = \langle t + 1, t - 1, \ln(4 - t^2) \rangle \) is continuous.
Theorem
A vector-valued function \( \mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle \) is continuous at \( t = a \) if and only if all of \( f \), \( g \), and \( h \) are continuous at \( t = a \).

Example

1. Determine the values of \( t \) for which \( \mathbf{r}(t) = \langle t + 1, t - 1, \ln(4 - t^2) \rangle \) is continuous.

2. Determine the values of \( t \) for which \( \mathbf{r}(t) = \langle \tan t, \cot t, e^{-t} \rangle \) is continuous.
**Recall:** for a real-valued function $f(t)$, the derivative of $f$ is defined as

$$
\lim_{h \to 0} \frac{f(t + h) - f(t)}{h}
$$

provided the limit exists.

For the sake of convenience later we will write this in the alternative form in which we replace $h$ with $\Delta t$.

$$
f'(t) = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}
$$
Derivatives of Vector-Valued Functions

Definition
The **derivative** \( r'(t) \) of the vector-valued function \( r(t) \) is

\[
r'(t) = \lim_{\Delta t \to 0} \frac{r(t + \Delta t) - r(t)}{\Delta t}
\]

provided the limit exists. When the limit exists for \( t = c \) we say that \( r \) is **differentiable** at \( t = c \).
Derivatives of Vector-Valued Functions

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provided the limit exists. When the limit exists for \( t = c \) we say that \( r \) is **differentiable** at \( t = c \).

Theorem
Let \( r(t) = \langle f(t), g(t), h(t) \rangle \) and suppose that the components \( f, g, \) and \( h \) are all differentiable for some value of \( t \). Then \( r \) is also differentiable at that value of \( t \) and

\[
r'(t) = \langle f'(t), g'(t), h'(t) \rangle.
\]
Examples

Find the derivatives of the following vector-valued functions provided they are differentiable.

1. \( \mathbf{r}(t) = \langle \sqrt{t}, 2 - t^3, \cos 2t \rangle \)

2. \( \mathbf{r}(t) = \langle \frac{t - 3}{t + 1}, te^{2t}, t^3 \tan t \rangle \)
Examples

Find the derivatives of the following vector-valued functions provided they are differentiable.

1. \( \mathbf{r}(t) = \langle \sqrt{t}, 2 - t^3, \cos 2t \rangle \)
   
   \[ \mathbf{r}'(t) = \langle \frac{1}{2\sqrt{t}}, -3t^2, -2\sin 2t \rangle \]

2. \( \mathbf{r}(t) = \langle \frac{t - 3}{t + 1}, te^{2t}, t^3 \tan t \rangle \)
Examples

Find the derivatives of the following vector-valued functions provided they are differentiable.

1. \( \mathbf{r}(t) = \langle \sqrt{t}, 2 - t^3, \cos 2t \rangle \)
   \[ \mathbf{r}'(t) = \langle \frac{1}{2\sqrt{t}}, -3t^2, -2 \sin 2t \rangle \]

2. \( \mathbf{r}(t) = \langle \frac{t - 3}{t + 1}, te^{2t}, t^3 \tan t \rangle \)
   \[ \mathbf{r}'(t) = \langle \frac{4}{(t + 1)^2}, (1 + 2t)e^{2t}, 3t^2 \tan t + t^3 \sec^2 t \rangle \]
Properties of the Derivative

Theorem

Suppose that \( r(t) \) and \( s(t) \) are differentiable vector-valued functions, \( f(t) \) is a differentiable scalar function, and \( c \) is a scalar constant. Then:

1. \( \frac{d}{dt}[r(t) + s(t)] = r'(t) + s'(t) \)
2. \( \frac{d}{dt}[cr(t)] = cr'(t) \)
3. \( \frac{d}{dt}[f(t)r(t)] = f'(t)r(t) + f(t)r'(t) \)
4. \( \frac{d}{dt}[r(t) \cdot s(t)] = r'(t) \cdot s(t) + r(t) \cdot s'(t) \)
5. \( \frac{d}{dt}[r(t) \times s(t)] = r'(t) \times s(t) + r(t) \times s'(t) \)
Smooth Curves

If a curve is traced out by the vector-valued function $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ for $t$ in the interval $[a, b]$ then the curve is \textbf{smooth} if $\mathbf{r}'$ is continuous on $[a, b]$ and $\mathbf{r}'(t) \neq \mathbf{0}$ except possibly at the endpoints.
Example (1 of 2)

Determine where the curve traced out by \( \mathbf{r}(t) = \langle t^4, t^3, t^2 \rangle \) is smooth.
Determine where the curve traced out by \( r(t) = \langle t^4, t^3, t^2 \rangle \) is smooth.

\[
r'(t) = \langle 4t^3, 3t^2, 2t \rangle
\]
Example (1 of 2)

Determine where the curve traced out by \( \mathbf{r}(t) = \langle t^4, t^3, t^2 \rangle \) is smooth.

\[
\mathbf{r}'(t) = \langle 4t^3, 3t^2, 2t \rangle
\]

**Note:** \( \mathbf{r}'(t) = 0 \) when \( t = 0 \).
Example (2 of 2)
Geometric Interpretation of $\mathbf{r}'(t)$

The derivative can be thought of as the tangent vector to the path.
Example

Suppose \( \mathbf{r}(t) = \langle 2 \cos t, \sin t, t \rangle \) and \( a = \pi/2 \).

1. Find the direction of the tangent vector for \( \mathbf{r}(t) \) at \( t = a \).
Example

Suppose \( \mathbf{r}(t) = \langle 2 \cos t, \sin t, t \rangle \) and \( a = \pi/2 \).

1. Find the direction of the tangent vector for \( \mathbf{r}(t) \) at \( t = a \).

\[
\mathbf{r}' \left( \frac{\pi}{2} \right) = \langle -2 \sin \frac{\pi}{2}, \cos \frac{\pi}{2}, 1 \rangle = \langle -2, 0, 1 \rangle
\]
Example

Suppose \( \mathbf{r}(t) = \langle 2 \cos t, \sin t, t \rangle \) and \( a = \pi/2 \).

1. Find the direction of the tangent vector for \( \mathbf{r}(t) \) at \( t = a \).

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\mathbf{r}' \left( \frac{\pi}{2} \right) = \langle -2 \sin \frac{\pi}{2}, \cos \frac{\pi}{2}, 1 \rangle = \langle -2, 0, 1 \rangle
\]

2. Find the parametric form of the tangent line to the curve generated by \( \mathbf{r}(t) \) at \( t = a \).
Example

Suppose \( \mathbf{r}(t) = \langle 2 \cos t, \sin t, t \rangle \) and \( a = \pi/2 \).

1. Find the direction of the tangent vector for \( \mathbf{r}(t) \) at \( t = a \).

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\mathbf{r}' \left( \frac{\pi}{2} \right) = \langle -2 \sin \frac{\pi}{2}, \cos \frac{\pi}{2}, 1 \rangle = \langle -2, 0, 1 \rangle
\]

2. Find the parametric form of the tangent line to the curve generated by \( \mathbf{r}(t) \) at \( t = a \).
Since \( \mathbf{r}(\pi/2) = \langle 0, 1, \pi/2 \rangle \) then

\[
\begin{align*}
x &= 0 + (-2)t = -2t \\
y &= 1 + (0)t = 1 \\
z &= \frac{\pi}{2} + t
\end{align*}
\]
Geometric Interpretation of $\mathbf{r}'(t)$
Orthogonality

Theorem
\[ \|r(t)\| \text{ is constant if and only if } r(t) \text{ and } r'(t) \text{ are orthogonal for all } t. \]
Orthogonality

Theorem
\[ \|r(t)\| \text{ is constant if and only if } r(t) \text{ and } r'(t) \text{ are orthogonal for all } t. \]

Proof.

- Suppose \( \|r(t)\| = C, \) a constant.
Orthogonality

Theorem
\[ \|r(t)\| \text{ is constant if and only if } r(t) \text{ and } r'(t) \text{ are orthogonal for all } t. \]

Proof.

▶ Suppose \( \|r(t)\| = C \), a constant.
▶ Suppose \( r(t) \cdot r'(t) = 0 \) for all \( t \).
Definition
The vector-valued function $\mathbf{R}(t)$ is an antiderivative of the vector-valued function $\mathbf{r}(t)$ whenever $\mathbf{R}'(t) = \mathbf{r}(t)$. If $\mathbf{R}(t)$ is any antiderivative of $\mathbf{r}(t)$, the indefinite integral of $\mathbf{r}(t)$ is defined to be $\int \mathbf{r}(t) \, dt = \mathbf{R}(t) + \mathbf{c}$ where $\mathbf{c}$ is an arbitrary constant vector.
Antiderivatives and Indefinite Integrals

Definition
The vector-valued function $R(t)$ is an antiderivative of the vector-valued function $r(t)$ whenever $R'(t) = r(t)$.

If $R(t)$ is any antiderivative of $r(t)$, the indefinite integral of $r(t)$ is defined to be

$$\int r(t) \, dt = R(t) + c$$

where $c$ is an arbitrary constant vector.
Example

Evaluate the following indefinite integrals.

1. \( \int \langle 2 \cos t, \sin t, 2t \rangle \ dt \)

2. \( \int \langle e^{-3t}, \sin 5t, t^{3/2} \rangle \ dt \)
Example

Evaluate the following indefinite integrals.

1. \[
\int \langle 2 \cos t, \sin t, 2t \rangle \, dt \\
= \langle 2 \sin t, -\cos t, t^2 \rangle + c
\]

2. \[
\int \langle e^{-3t}, \sin 5t, t^{3/2} \rangle \, dt
\]
Example

Evaluate the following indefinite integrals.

1. \[ \int \langle 2 \cos t, \sin t, 2t \rangle \, dt \]
   \[ \int \langle 2 \cos t, \sin t, 2t \rangle \, dt = \langle 2 \sin t, -\cos t, t^2 \rangle + c \]

2. \[ \int \langle e^{-3t}, \sin 5t, t^{3/2} \rangle \, dt \]
   \[ \int \langle e^{-3t}, \sin 5t, t^{3/2} \rangle \, dt = \left\langle -\frac{1}{3} e^{-3t}, -\frac{1}{5} \cos 5t, \frac{2}{5} t^{5/2} \right\rangle + c \]
Definite Integrals

Definition
For the vector-valued function \( \mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle \), we define the definite integral of \( \mathbf{r}(t) \) on the interval \([a, b]\) by

\[
\int_a^b \mathbf{r}(t) \, dt = \int_a^b \langle f(t), g(t), h(t) \rangle \, dt
\]

\[
= \left\langle \int_a^b f(t) \, dt, \int_a^b g(t) \, dt, \int_a^b h(t) \, dt \right\rangle
\]

Theorem
Suppose that \( R(t) \) is an antiderivative of \( \mathbf{r}(t) \) on the interval \([a, b]\), then

\[
\int_a^b \mathbf{r}(t) \, dt = R(b) - R(a).
\]
Definite Integrals

Definition
For the vector-valued function \( \mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle \), we define the **definite integral** of \( \mathbf{r}(t) \) on the interval \([a, b]\) by

\[
\int_a^b \mathbf{r}(t) \, dt = \int_a^b \langle f(t), g(t), h(t) \rangle \, dt
\]

\[
= \left\langle \int_a^b f(t) \, dt, \int_a^b g(t) \, dt, \int_a^b h(t) \, dt \right\rangle
\]

Theorem
*Suppose that \( \mathbf{R}(t) \) is an antiderivative of \( \mathbf{r}(t) \) on the interval \([a, b]\), then*

\[
\int_a^b \mathbf{r}(t) \, dt = \mathbf{R}(b) - \mathbf{R}(a).
\]
Evaluate the following definite integral.

$$\int_0^{\pi/2} \langle 2 \cos t, \sin t, 2t \rangle \, dt$$
Evaluate the following definite integral.

\[
\int_0^{\pi/2} \langle 2 \cos t, \sin t, 2t \rangle \, dt
\]

\[
\int_0^{\pi/2} \langle 2 \cos t, \sin t, 2t \rangle \, dt = \langle 2 \sin t, -\cos t, t^2 \rangle \bigg|_0^{\pi/2}
\]

\[
= \langle 2, 1, \frac{\pi^2}{4} \rangle
\]
Evaluate the following definite integral.

\[
\int_0^1 \langle e^{-3t}, \sin 5t, t^{3/2} \rangle \, dt
\]
Evaluate the following definite integral.

\[ \int_0^1 \langle e^{-3t}, \sin 5t, t^{3/2} \rangle \, dt \]

\[ \int_0^1 \langle e^{-3t}, \sin 5t, t^{3/2} \rangle \, dt = \left. \left\langle -\frac{1}{3}e^{-3t}, -\frac{1}{5}\cos 5t, \frac{2}{5}t^{5/2} \right\rangle \right|_0^1 \]

\[ = \left\langle \frac{1}{3}(1 - e^{-3}), \frac{1}{5}(1 - \cos 5), \frac{2}{5} \right\rangle \]
Homework

- Read Section 11.2.
- Exercises: 1–49 odd.