Cylindrical Coordinates

MATH 311, *Calculus III*

J. Robert Buchanan

Department of Mathematics

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Cylindrical Coordinate System

The cylindrical coordinate system is an alternative to the Cartesian coordinate system for locating points in $\mathbb{R}^3$. 
Example

Cylinder: \( x^2 + y^2 = c^2 \)
Example

Cone: \( z = \sqrt{x^2 + y^2} \).
Example

Ellipsoid: \( \frac{x^2}{a^2} + \frac{y^2}{a^2} + z^2 = 1 \).

\[ z^2 = 1 - \frac{r^2}{a^2} \]
Example

Hyperboloid of 1 Sheet: \( x^2 + y^2 - z^2 = a^2 \). 

\[
r^2 - z^2 = a^2
\]
In cylindrical coordinates $dV = r\,dr\,d\theta\,dz$ and thus the triple integral becomes

$$\int\int\int_Q f(r, \theta, z)\,dV = \int_\alpha^{\beta} \int_{g_1(\theta)}^{g_2(\theta)} \int_{k_1(r, \theta)}^{k_2(r, \theta)} f(r, \theta, z)\,r\,dz\,dr\,d\theta.$$
Example

Convert the following triple integral to cylindrical coordinates and evaluate it.

\[
\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{2} (x^2 + y^2) \, dz \, dy \, dx
\]
Solution

\[
\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{-\sqrt{x^2+y^2}}^{\sqrt{x^2+y^2}} (x^2 + y^2) \, dz \, dy \, dx
\]

\[
= \int_{0}^{2\pi} \int_{0}^{2} \int_{0}^{r} (r^2) r \, dz \, dr \, d\theta
\]

\[
= 2\pi \int_{0}^{2} \int_{0}^{r} r^3 \, dz \, dr
\]

\[
= 2\pi \int_{0}^{2} r^3 (2 - r) \, dr
\]

\[
= 2\pi \left( \frac{1}{2} r^4 - \frac{1}{5} r^5 \right) \bigg|_{0}^{2}
\]

\[
= 2\pi \left( 8 - \frac{32}{5} \right) = \frac{16\pi}{5}
\]
Example

Use the limits of integration to picture the solid region \( Q \) for the triple integral.

\[
\sqrt{x^2 + y^2} \leq z \leq 2
\]

\[-\sqrt{4 - x^2} \leq y \leq \sqrt{4 - x^2}\]

\[-2 \leq x \leq 2\]
Example: Mass and Density

A solid region $Q$ lies within the cylinder $x^2 + y^2 = 1$, below the plane $z = 4$ and above the paraboloid $z = 1 - x^2 - y^2$. The density at any point in $Q$ is proportional to its distance from the axis of the cylinder. Find the mass of $Q$. 
Solution

\[ m = \iiint_Q \rho(r, \theta, z) \, dV = \int_0^{2\pi} \int_0^1 \int_{1-r^2}^4 (r) \, r \, dz \, dr \, d\theta \]

\[ = 2\pi \int_0^1 \int_{1-r^2}^4 r^2 \, dz \, dr \]

\[ = 2\pi \int_0^1 r^2 \left. \left( z \right) \right|_{1-r^2}^4 \, dr = 2\pi \int_0^1 4r^2 - r^2(1 - r^2) \, dr \]

\[ = 2\pi \int_0^1 3r^2 + r^4 \, dr \]

\[ = 2\pi \left( r^3 + \frac{1}{5} r^5 \right) \bigg|_0^1 \]

\[ = \frac{12\pi}{5} \]
Example

Find the volume of the solid region that the cylinder $r = a \cos \theta$ cuts out of the sphere of radius $a$ centered at the origin.
Solution (1 of 2)

- Equation of the sphere (Cartesian coordinates):
  \[ x^2 + y^2 + z^2 = a^2. \]

- Equation of the sphere (cylindrical coordinates):
  \[ r^2 + z^2 = a^2. \]

- Limits of integration:
  - \[ -\sqrt{a^2 - r^2} \leq z \leq \sqrt{a^2 - r^2} \]
  - \[ 0 \leq r \leq a \cos \theta \]
  - \[ 0 \leq \theta \leq \pi \]
\[
V = \iiint_Q 1 \, dV
\]
\[
= \int_{-\pi/2}^{\pi/2} \int_0^{\acos \theta} \int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} r \, dz \, dr \, d\theta
\]
\[
= \int_{-\pi/2}^{\pi/2} \int_0^{\acos \theta} 2r \sqrt{a^2 - r^2} \, dr \, d\theta
\]
\[
= \int_{-\pi/2}^{\pi/2} \left( -\frac{2}{3} (a^2 - r^2)^{3/2} \right) \bigg|_0^{\acos \theta} d\theta
\]
\[
= \frac{2a^3}{3} \int_{-\pi/2}^{\pi/2} 1 - \sin^3 \theta \, d\theta
\]
\[
= \frac{2a^3}{3} \int_{-\pi/2}^{\pi/2} 1 \, d\theta = \frac{2\pi a^3}{3}
\]
Homework

- Read Section 13.6.
- Exercises: 1–45 odd