Double Integrals
MATH 311, Calculus III

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Riemann Integral

Recall: the Riemann integral of a function $f(x)$ on the interval $[a, b]$ is

$$\int_{a}^{b} f(x) \, dx = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(w_i) \Delta x_i$$

where

$P$: is a partition of $[a, b]$, i.e. $P = \{x_0, x_1, \ldots, x_n\}$ with $a = x_0 \leq x_1 \leq \cdots \leq x_{i-1} \leq x_i \leq \cdots \leq x_n = b$.

$\Delta x_i$: is the width of the $i$th subinterval

$$\Delta x_i = x_i - x_{i-1}.$$ 

$w_i$: is a number from the $i$th subinterval, $x_{i-1} \leq w_i \leq x_i$.

$\|P\|$: is the norm of the partition,

$$\|P\| = \max_{1 \leq i \leq n} \{\Delta x_i\}.$$
Illustration
Double Integrals Over a Rectangle

**Task:** For a continuous function \( f(x, y) \geq 0 \) defined on the rectangle \( R = \{(x, y) \mid a \leq x \leq b, c \leq y \leq d\} \), find the volume under the surface \( z = f(x, y) \) and above the xy-plane.

**Approach:**

1. partition \([a, b]\) and \([c, d]\)

   \[
   a = x_0 \leq x_1 \leq \cdots \leq x_{i-1} \leq x_i \leq \cdots \leq x_n = b \\
   c = y_0 \leq y_1 \leq \cdots \leq y_{j-1} \leq y_j \leq \cdots \leq y_m = d
   \]

2. define \( \Delta A_{ij} = (x_i - x_{i-1})(y_j - y_{j-1}) \)

3. in each rectangle of the form \( R_{ij} = \{(x, y) : x_{i-1} \leq x \leq x_i, y_{j-1} \leq y \leq y_j\} \)

   select an ordered pair \((u_{ij}, v_{ij})\).

4. define the Riemann sum

   \[
   V = \sum_{i=1}^{n} \sum_{j=1}^{m} f(u_{ij}, v_{ij}) \Delta A_{ij}. 
   \]
Rectangular Region
Riemann Sum
Approximate, using a Riemann sum, the volume under the surface generated by

\[ f(x, y) = 10 - (x - 2)^2 - (y - 2)^2 \]

over the rectangle \( R = \{(x, y) : 1 \leq x \leq 3, 1 \leq y \leq 4\} \).
Choose \( n = 5 \) and \( m = 7 \).

Define \( \Delta x = (3 - 1)/5 = 2/5 \) and \( \Delta y = (4 - 1)/7 = 3/7 \), then \( \Delta A = (\Delta x)(\Delta y) = 6/35 \).

Define \( x_i = 1 + 2i/5 \) for \( i = 0, 1, \ldots, 5 \) and \( y_j = 1 + 3j/7 \) for \( j = 0, 1, \ldots, 7 \).

Choose \( u_i = 1 + 2i/5 \) for \( i = 1, \ldots, 5 \) and \( v_j = 1 + 3j/7 \) for \( j = 1, \ldots, 7 \).

\[
V \approx \frac{6}{35} \sum_{i=1}^{5} \sum_{j=1}^{7} f \left( 1 + \frac{2i}{5}, 1 + \frac{3j}{7} \right) \\
\approx 50.3706
\]
Finding the Exact Volume

Think of the **norm** of the partition of $R$ as the length of the longest diagonal in the partition of $R$.

**Definition**
For any function $f(x, y)$ defined on the rectangle $R = \{(x, y) \mid a \leq x \leq b, c \leq y \leq d\}$ the **double integral** of $f$ over $R$ is

$$\int\int_R f(x, y) \, dA = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(u_i, v_i) \Delta A_i$$

provided the limit exists and is the same for every choice of evaluation points $(u_i, v_i)$ in $R_i$. Under these conditions we say that $f$ is **integrable** over $R$. 
Theorem (Fubini’s Theorem)

Suppose that $f$ is integrable over the rectangle $R = \{(x, y) \mid a \leq x \leq b, c \leq y \leq d\}$. Then we can write the double integral over $R$ as either of the iterated integrals:

$$
\int \int_{R} f(x, y) \, dA = \int_{a}^{b} \int_{c}^{d} f(x, y) \, dy \, dx = \int_{c}^{d} \int_{a}^{b} f(x, y) \, dx \, dy.
$$
Evaluate the following double integral.

\[ \int_{1}^{3} \int_{1}^{4} \left( 10 - (x - 2)^2 - (y - 2)^2 \right) \, dy \, dx \]
Solution

\[ \int_1^3 \int_1^4 \left( 10 - (x - 2)^2 - (y - 2)^2 \right) \, dy \, dx \]

\[ = \int_1^3 \left( 10y - (x - 2)^2 y - \frac{1}{3}(y - 2)^3 \right) \bigg|_1^4 \, dx \]

\[ = \int_1^3 \left[ \left( 40 - 4(x - 2)^2 - \frac{8}{3} \right) - \left( 10 - (x - 2)^2 + \frac{1}{3} \right) \right] \, dx \]

\[ = \int_1^3 \left( 27 - 3(x - 2)^2 \right) \, dx \]

\[ = 27x - (x - 2)^3 \bigg|_1^3 \]

\[ = (81 - 1) - (27 + 1) = 52 \]
Evaluate the following double integral.

\[
\int_{-1}^{1} \int_{0}^{2} 4xe^{2y} \, dx \, dy
\]
Solution

\[ \int_{-1}^{1} \int_{0}^{2} 4xe^{2y} \, dx \, dy = \int_{-1}^{1} \left. 2x^2e^{2y} \right|_{0}^{2} \, dy \]

\[ = \int_{-1}^{1} 8e^{2y} \, dy \]

\[ = \left. 4e^{2y} \right|_{-1}^{1} \]

\[ = 4e^2 - 4e^{-2} \]
An **inner partition** consists of rectangular elements which lie completely inside of $R$. 
Definition
For any function \( f(x, y) \) defined on a bounded region \( R \subset \mathbb{R}^2 \), the \textbf{double integral} of \( f \) over \( R \) is

\[
\iint_R f(x, y) \, dA = \lim_{\|P\| \to 0} \sum_{i=1}^n f(u_i, v_i) \Delta A_i
\]

provided the limit exists and is the same for every choice of evaluation points \( (u_i, v_i) \) in \( R \). Under these conditions we say that \( f \) is \textbf{integrable} over \( R \).
Theorem

Suppose that $f(x, y)$ is continuous on the region $R$ defined by

$R = \{(x, y) \mid a \leq x \leq b, \ g_1(x) \leq y \leq g_2(x)\}$,

for continuous functions $g_1$ and $g_2$, where $g_1(x) \leq g_2(x)$ for all $a \leq x \leq b$.

Then

$$\int \int_{R} f(x, y) \, dA = \int_{a}^{b} \int_{g_1(x)}^{g_2(x)} f(x, y) \, dy \, dx.$$
Example (1 of 2)
Evaluate the iterated integral:

\[
\int_{-1}^{1} \int_{0}^{1-x^2} (3x^2 + 2y) \, dy \, dx
\]
Example (2 of 2)

\[
\int_{-1}^{1} \int_{0}^{1-x^2} (3x^2 + 2y) \, dy \, dx = \int_{-1}^{1} \left( 3x^2y + y^2 \right) \bigg|_{0}^{1-x^2} \, dx
\]

\[
= \int_{-1}^{1} \left( 3x^2(1 - x^2) + (1 - x^2)^2 \right) \, dx
\]

\[
= \int_{-1}^{1} \left( 1 + x^2 - 2x^4 \right) \, dx
\]

\[
= \left( x + \frac{1}{3}x^3 - \frac{2}{5}x^5 \right) \bigg|_{-1}^{1}
\]

\[
= \left( 1 + \frac{1}{3} - \frac{2}{5} \right) - \left( -1 - \frac{1}{3} + \frac{2}{5} \right)
\]

\[
= \frac{28}{15}
\]
Theorem

Suppose that \( f(x, y) \) is continuous on the region \( R \) defined by
\[ R = \{(x, y) \mid c \leq y \leq d, \ h_1(y) \leq x \leq h_2(y)\}, \]
for continuous functions \( h_1 \) and \( h_2 \), where \( h_1(y) \leq h_2(y) \) for all \( c \leq y \leq d \).

Then
\[
\int\int_R f(x, y) \, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) \, dx \, dy.
\]
Example (1 of 2)
Evaluate the iterated integral:

\[
\int_{0}^{1} \int_{0}^{y^2} \frac{3}{4 + y^3} \, dx \, dy
\]
Example (2 of 2)

\[
\int_0^1 \int_0^{y^2} \frac{3}{4 + y^3} \, dx \, dy = \int_0^1 \frac{3x}{4 + y^3} \bigg|_0^{y^2} \, dy
\]

\[
= \int_0^1 \frac{3y^2}{4 + y^3} \, dy
\]

\[
= \ln |4 + y^3| \bigg|_0^1
\]

\[
= \ln 5 - \ln 4
\]
Changing Order of Integration

Sometimes a double integral may be easier to evaluate when integrated in a specific order.

**Example**

Determine the most convenient order of integration for, and evaluate the following double integral.

\[ \int_{0}^{1} \int_{\sqrt{y}}^{1} \cos(x^3) \, dx \, dy \]
Determining the Order of Integration

\[
\sqrt{y} = x \quad \iff \quad y = x^2
\]
Changing the Order of Integration

Given
\[
\int_0^1 \int_{\sqrt{y}}^1 \cos(x^3) \, dx \, dy
\]

it would be difficult to integrate with respect to \( x \) first, so change the order of integration, keeping the region being integrated over the same.

\[
\int_0^1 \int_{\sqrt{y}}^1 \cos(x^3) \, dx \, dy = \int_0^1 \int_0^{x^2} \cos(x^3) \, dy \, dx
\]

\[
= \int_0^1 x^2 \cos(x^3) \, dx
\]

\[
= \frac{1}{3} \sin(x^3) \bigg|_0^1
\]

\[
= \frac{1}{3} \sin(1)
\]
Properties of the Double Integral

Theorem

Let \( f(x, y) \) and \( g(x, y) \) be integrable over the region \( R \subset \mathbb{R}^2 \) and let \( c \) be any constant. Then, the following hold:

1. \[ \iint_R c f(x, y) \, dA = c \iint_R f(x, y) \, dA, \]
2. \[ \iint_R [f(x, y) + g(x, y)] \, dA = \iint_R f(x, y) \, dA + \iint_R g(x, y) \, dA, \]
3. if \( R = R_1 \cup R_2 \), where \( R_1 \) and \( R_2 \) are non-overlapping regions, then
   \[ \iint_R f(x, y) \, dA = \iint_{R_1} f(x, y) \, dA + \iint_{R_2} f(x, y) \, dA \]
Homework

- Read Section 13.1.
- Exercises: 1, 5–31 odd, 37–53 odd