Constrained Optimization and Lagrange Multipliers
MATH 311, *Calculus III*

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In the previous section we found the local or absolute extrema of a function either on the entire domain of the function or on a bounded region.

Now we will look for extrema which satisfy some side condition(s) known as **constraint(s)**.
Example

Find the point on the parabola \( y = x^2 + 3x + 2 \) closest to the origin.

\[
f(x, y) = x^2 + (x^2 + 3x + 2)^2
\]
Geometry (1 of 2)

Note:

- At the minimum distance from the origin the parabola and the circle are tangent.
- The normals to the parabola and the circle are parallel at the point of tangency.
- The gradient is always normal to the curve.
Geometry (2 of 2)

parabola: \[ x^2 + 3x + 2 - y = 0 \]
circle: \[ x^2 + y^2 = r^2 \]

Gradients:
\[
\nabla(x^2 + y^2) = \lambda \nabla(x^2 + 3x + 2 - y)
\]
\[
\langle 2x, 2y \rangle = \lambda \langle 2x + 3, -1 \rangle
\]

Equivalent system of equations:
\[
2x = \lambda(2x + 3)
\]
\[
2y = -\lambda
\]
\[
0 = x^2 + 3x + 2 - y
\]
Solution

\[ y = x^2 + 3x + 2 \]
\[ \lambda = -2(x^2 + 3x + 2) \]
\[ 0 = -2(x^2 + 3x + 2)(2x + 3) - 2x \]
\[ x \approx -0.682817 \]

which implies \( y \approx 0.417788. \)
Method of Lagrange Multipliers (1 of 2)

**Problem:** find the extreme values of \( f(x, y, z) \) subject to the constraint \( g(x, y, z) = 0 \).

**Solution:**

- Suppose \( f \) has an extremum at \( (x_0, y_0, z_0) \) on the surface \( S \) defined by \( g(x, y, z) = 0 \).
- Let \( C \) be a curve traced out by the vector-valued function \( \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle \) such that \( \mathbf{r}(t_0) = \langle x_0, y_0, z_0 \rangle \).
- Define \( h(t) = f(x(t), y(t), z(t)) \), then at the extremum \( h'(t_0) = 0 \).
Method of Lagrange Multipliers (2 of 2)

\[
0 = h'(t_0) \\
= f_x(x(t_0), y(t_0), z(t_0))x'(t_0) + f_y(x(t_0), y(t_0), z(t_0))y'(t_0) \\
+ f_z(x(t_0), y(t_0), z(t_0))z'(t_0) \\
= \langle f_x(x_0, y_0, z_0), f_y(x_0, y_0, z_0), f_z(x_0, y_0, z_0) \rangle \cdot \langle x'(t_0), y'(t_0), z'(t_0) \rangle \\
= \nabla f(x_0, y_0, z_0) \cdot r'(t_0)
\]

Thus \( \nabla f(x_0, y_0, z_0) \) is orthogonal to \( r'(t_0) \).
0 = h'(t_0)
= f_x(x(t_0), y(t_0), z(t_0))x'(t_0) + f_y(x(t_0), y(t_0), z(t_0))y'(t_0)
+ f_z(x(t_0), y(t_0), z(t_0))z'(t_0)
= \langle f_x(x_0, y_0, z_0), f_y(x_0, y_0, z_0), f_z(x_0, y_0, z_0) \rangle \cdot \langle x'(t_0), y'(t_0), z'(t_0) \rangle
= \nabla f(x_0, y_0, z_0) \cdot r'(t_0)

Thus \nabla f(x_0, y_0, z_0) is orthogonal to r'(t_0).

Since r(t) is arbitrary, \nabla f(x_0, y_0, z_0) is orthogonal to S and hence parallel to \nabla g(x_0, y_0, z_0).

\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)
Theorem
Suppose that \(f(x, y, z)\) and \(g(x, y, z)\) are functions with continuous first partial derivatives and \(\nabla g(x, y, z) \neq \mathbf{0}\) on the surface \(g(x, y, z) = 0\). Suppose that either

1. the minimum value of \(f(x, y, z)\) subject to the constraint \(g(x, y, z) = 0\) occurs at \((x_0, y_0, z_0)\); or

2. the maximum value of \(f(x, y, z)\) subject to the constraint \(g(x, y, z) = 0\) occurs at \((x_0, y_0, z_0)\).

Then \(\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)\), for some constant \(\lambda\) (called a Lagrange multiplier).
Equivalent Set of Equations

\[ f_x(x, y, z) = \lambda g_x(x, y, z) \]
\[ f_y(x, y, z) = \lambda g_y(x, y, z) \]
\[ f_z(x, y, z) = \lambda g_z(x, y, z) \]
\[ g(x, y, z) = 0 \]
Equivalent Set of Equations

\begin{align*}
  f_x(x, y, z) &= \lambda g_x(x, y, z) \\
  f_y(x, y, z) &= \lambda g_y(x, y, z) \\
  f_z(x, y, z) &= \lambda g_z(x, y, z) \\
  g(x, y, z) &= 0
\end{align*}

For functions of two variables this becomes:

\begin{align*}
  f_x(x, y) &= \lambda g_x(x, y) \\
  f_y(x, y) &= \lambda g_y(x, y) \\
  g(x, y) &= 0
\end{align*}
Example (1 of 3)

Find the extreme values of \( f(x, y) = 2x^3y \) subject to \( x^2 + y^2 = 4 \).
Example (3 of 3)

System of equations:

\[ 6x^2y = 2\lambda x \]
\[ 2x^3 = 2\lambda y \]
\[ x^2 + y^2 = 4 \]

Cases:

- If \( x = 0 \) then \( \lambda = 0 \) and \( y = \pm 2 \).
- If \( x \neq 0 \) then \( x^2 = 3y^2 \) and \( y = \pm 1 \) and \( x = \pm \sqrt{3} \).

Maximum \( f(\pm \sqrt{3}, \pm 1) = 6\sqrt{3} \), minimum \( f(\pm \sqrt{3}, \mp 1) = -6\sqrt{3} \).
Find the extrema of $f(x, y) = 4xy$ subject to $x^2 + 4y^2 \leq 8$. 
Example (2 of 3)
System of equations:

\[ 4y = 2\lambda x \]
\[ 4x = 8\lambda y \]
\[ x^2 + 4y^2 = 8 \]

Cases:

- If \( \lambda = 1 \) then \((x, y) = (\pm 2, \pm 1)\).
- If \( \lambda = -1 \) then \((x, y) = (\pm 2, \mp 1)\).

Maximum \( f(\pm 2, \pm 1) = 8 \), minimum \( f(\pm 2, \mp 1) = -8 \).

There is a critical point at \((x, y) = (0, 0)\) but this is a saddle point according to the Second Derivatives Test.
Two Equality Constraints (1 of 3)

Maximize \( f(x, y, z) = 3x + y + 2z \) subject to \( y^2 + z^2 = 1 \) and \( x + y - z = 0 \).
Two Equality Constraints (2 of 3)

\[ \nabla (3x + y + 2z) = \lambda \nabla (y^2 + z^2 - 1) + \mu \nabla (x + y - z) \]

This is equivalent to the system of equations:

\[\begin{align*}
3 &= \mu \\
1 &= 2\lambda y + \mu \\
2 &= 2\lambda z - \mu \\
1 &= y^2 + z^2 \\
0 &= x + y - z
\end{align*}\]
The five equations can be reduced to:

\[-1 = \lambda y \]
\[5 = 2\lambda z \]
\[1 = y^2 + z^2 \]
\[0 = x + y - z \]

The first 3 equations yield \( \lambda = \pm \sqrt{29}/2 \), then

\((x, y, z) = (\pm 7/\sqrt{29}, \mp 2/\sqrt{29}, \pm 5/\sqrt{29})\).

Maximum: \( f(7/\sqrt{29}, -2/\sqrt{29}, 5/\sqrt{29}) = \sqrt{29} \) and minimum:

\( f(-7/\sqrt{29}, 2/\sqrt{29}, -5/\sqrt{29}) = -\sqrt{29} \).
Homework

- Read Section 12.8.
- Exercises: 1, 5, 9, 13, 17, 21, 25, 29, 41