Limits and Continuity
MATH 311, *Calculus III*

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Fall 2015
Limits of Functions of Two Variables

Definition
Let \( f \) be defined on the interior of a circle centered at the point \((a, b)\), except possibly at \((a, b)\) itself. We say that

\[
\lim_{(x,y) \to (a,b)} f(x, y) = L
\]

if for every \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that \( |f(x, y) - L| < \epsilon \)
whenever \( 0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta \).
Properties of the Limit

**Theorem**

*Suppose* \( \lim_{(x,y) \to (a,b)} f(x, y) \) and \( \lim_{(x,y) \to (a,b)} g(x, y) \) both exist, then

1. \( \lim_{(x,y) \to (a,b)} [f(x, y) \pm g(x, y)] = \lim_{(x,y) \to (a,b)} f(x, y) \pm \lim_{(x,y) \to (a,b)} g(x, y) \)

2. \( \lim_{(x,y) \to (a,b)} [f(x, y)g(x, y)] = \left[ \lim_{(x,y) \to (a,b)} f(x, y) \right] \left[ \lim_{(x,y) \to (a,b)} g(x, y) \right] \)

3. \( \lim_{(x,y) \to (a,b)} \frac{f(x, y)}{g(x, y)} = \frac{\lim_{(x,y) \to (a,b)} f(x, y)}{\lim_{(x,y) \to (a,b)} g(x, y)} \), provided \( \lim_{(x,y) \to (a,b)} g(x, y) \neq 0 \),
Example

Evaluate the following limit, if it exists.

$$\lim_{(x,y) \to (2,-1)} \frac{x + y}{x^2 - 2xy} = \frac{2 - 1}{2^2 - 2(2)(-1)} = \frac{1}{8}$$
Evaluate the following limit, if it exists.

\[
\lim_{(x,y) \to (2,-1)} \frac{x + y}{x^2 - 2xy}
\]

\[
\lim_{(x,y) \to (2,-1)} \frac{x + y}{x^2 - 2xy} = \frac{\lim_{(x,y) \to (2,-1)}(x + y)}{\lim_{(x,y) \to (2,-1)}(x^2 - 2xy)}
\]

\[
= \frac{(2 + (-1))}{2^2 - 2(2)(-1)}
\]

\[
= \frac{1}{8}
\]
The limit of \( f(x, y) \) as \((x, y)\) approaches \((a, b)\) must be the same \textbf{no matter which path the approach takes.}  

If \( f(x, y) \) approaches \( L_1 \) as \((x, y)\) approached \((a, b)\) along a path \( P_1 \) and \( f(x, y) \) approaches \( L_2 \) as \((x, y)\) approached \((a, b)\) along a path \( P_2 \) and \( L_1 \neq L_2 \), then

\[
\lim_{(x, y) \to (a, b)} f(x, y) \text{ does not exist.}
\]

The simplest paths to check are horizontal and vertical lines.
Show the following limit does not exist.

$$\lim_{(x,y) \to (0,0)} \frac{x^2 - y^2}{x^2 + y^2}$$
Example (2 of 2)

Suppose the path to (0, 0) is parameterized as

\[ x(t) = 0 \]
\[ y(t) = t \]

and consider the limit

\[ \lim_{t \to 0} \frac{[x(t)]^2 - [y(t)]^2}{[x(t)]^2 + [y(t)]^2} = \lim_{t \to 0} \frac{-t^2}{t^2} = -1. \]
Example (2 of 2)

Suppose the path to \((0, 0)\) is parameterized as

\[
\begin{align*}
x(t) &= 0 \\
y(t) &= t
\end{align*}
\]

and consider the limit

\[
\lim_{t \to 0} \frac{[x(t)]^2 - [y(t)]^2}{[x(t)]^2 + [y(t)]^2} = \lim_{t \to 0} \frac{-t^2}{t^2} = -1.
\]

Now suppose the path to \((0, 0)\) is parameterized as

\[
\begin{align*}
x(t) &= t \\
y(t) &= 0
\end{align*}
\]

and consider the limit

\[
\lim_{t \to 0} \frac{[x(t)]^2 - [y(t)]^2}{[x(t)]^2 + [y(t)]^2} = \lim_{t \to 0} \frac{t^2}{t^2} = 1.
\]

Thus the original limit does not exist.
Show the following limit does not exist.

$$\lim_{{(x,y) \to (0,0)}} \frac{xy}{x^2 + y^2}$$
Example (2 of 2)

Suppose the path to \((0,0)\) is parameterized as

\[
\begin{align*}
  x(t) &= t \\
  y(t) &= t
\end{align*}
\]

and consider the limit

\[
\lim_{t \to 0} \frac{[x(t)][y(t)]}{[x(t)]^2 + [y(t)]^2} = \lim_{t \to 0} \frac{t^2}{2t^2} = \frac{1}{2}.
\]

Thus the original limit does not exist.
Example (2 of 2)

Suppose the path to \((0, 0)\) is parameterized as

\[
\begin{align*}
x(t) &= t \\
y(t) &= t
\end{align*}
\]

and consider the limit

\[
\lim_{t \to 0} \frac{[x(t)][y(t)]}{[x(t)]^2 + [y(t)]^2} = \lim_{t \to 0} \frac{t^2}{2t^2} = \frac{1}{2}.
\]

Now suppose the path to \((0, 0)\) is parameterized as

\[
\begin{align*}
x(t) &= t \\
y(t) &= -t
\end{align*}
\]

and consider the limit

\[
\lim_{t \to 0} \frac{[x(t)][y(t)]}{[x(t)]^2 + [y(t)]^2} = \lim_{t \to 0} \frac{-t^2}{2t^2} = -\frac{1}{2}.
\]

Thus the original limit does not exist.
Example (1 of 2)

Show the following limit does not exist.

$$\lim_{(x,y) \to (0,0)} \frac{xy^2}{x^2 + y^4}$$
Example (2 of 2)

Suppose the path to $(0, 0)$ is parameterized as

\[ x(t) = t^2 \]
\[ y(t) = t \]

and consider the limit

\[
\lim_{t \to 0} \frac{[x(t)][y(t)]^2}{[x(t)]^2 + [y(t)]^4} = \lim_{t \to 0} \frac{t^4}{2t^4} = \frac{1}{2}.
\]
Example (2 of 2)

Suppose the path to \((0, 0)\) is parameterized as

\[
\begin{align*}
  x(t) &= t^2 \\
  y(t) &= t
\end{align*}
\]

and consider the limit

\[
\lim_{t \to 0} \frac{[x(t)][y(t)]^2}{[x(t)]^2 + [y(t)]^4} = \lim_{t \to 0} \frac{t^4}{2t^4} = \frac{1}{2}.
\]

Now suppose the path to \((0, 0)\) is parameterized as

\[
\begin{align*}
  x(t) &= -t^2 \\
  y(t) &= t
\end{align*}
\]

and consider the limit

\[
\lim_{t \to 0} \frac{[x(t)][y(t)]^2}{[x(t)]^2 + [y(t)]^4} = \lim_{t \to 0} \frac{-t^4}{2t^4} = -\frac{1}{2}.
\]

Thus the original limit does not exist.
Theorem

Suppose that $|f(x, y) - L| \leq g(x, y)$ for all $(x, y)$ in the interior of some circle centered at $(a, b)$, except possibly at $(a, b)$. If $\lim_{(x, y) \to (a, b)} g(x, y) = 0$, then $\lim_{(x, y) \to (a, b)} f(x, y) = L$. 
Find the following limit, if it exists.

\[ \lim_{(x,y) \to (0,0)} \frac{3x^2y}{x^2 + y^2} \]
Example (2 of 3)

Note that if \((x, y) \neq (0, 0)\) then

\[
\left| \frac{3x^2y}{x^2 + y^2} - 0 \right| \leq \left| \frac{3x^2y}{x^2} \right| = 3|y|.
\]
Example (2 of 3)

Note that if \((x, y) \neq (0, 0)\) then

\[
\left| \frac{3x^2y}{x^2 + y^2} - 0 \right| \leq \left| \frac{3x^2y}{x^2} \right| = 3|y|.
\]

Since

\[
\lim_{(x, y) \to (0, 0)} 3|y| = \lim_{y \to 0} 3|y| = 0
\]

then

\[
\lim_{(x, y) \to (0, 0)} \frac{3x^2y}{x^2 + y^2} = 0.
\]
Example (3 of 3)

A polar coordinate transformation can also sometimes be used to establish a limit.

\[
x(\theta) = r \cos \theta \\
y(\theta) = r \sin \theta
\]
Example (3 of 3)

A polar coordinate transformation can also sometimes be used to establish a limit.

\[ x(\theta) = r \cos \theta \]
\[ y(\theta) = r \sin \theta \]

Consider the limit:

\[
\lim_{r \to 0} \frac{3[x(\theta)]^2 y(\theta)}{[x(\theta)]^2 + [y(\theta)]^2} = \lim_{r \to 0} \frac{3r^3 \cos^2 \theta \sin \theta}{r^2 \cos^2 \theta + r^2 \sin^2 \theta}
\]
\[
= \lim_{r \to 0} \frac{3r^3 \cos^2 \theta \sin \theta}{r^2} = \lim_{r \to 0} 3r \cos^2 \theta \sin \theta = 0
\]

independent of \( \theta \).
Definition

Suppose that \( f(x, y) \) is defined in the interior of a circle centered at the point \((a, b)\). We say that \( f \) is *continuous* at \((a, b)\) if

\[
\lim_{(x,y) \to (a,b)} f(x, y) = f(a, b).
\]

If \( f(x, y) \) is not continuous at \((a, b)\), then we call \((a, b)\) a *discontinuity* of \( f \).
Open and Closed Disks

Open: \((x-a)^2 + (y-b)^2 < \delta^2\)

Closed: \((x-a)^2 + (y-b)^2 \leq \delta^2\)
Interior Points

Definition

A point \((a, b)\) in a region \(R\) in the plane is an **interior point** of \(R\) if there is an open disk of some positive radius centered at \((a, b)\) which lies completely within \(R\).
Boundary Points

Definition
A point \((a, b)\) in a region \(R\) in the plane is a **boundary point** of \(R\) if every open disk of positive radius centered at \((a, b)\) contains points in \(R\) and points outside \(R\).
Open and Closed Regions

Definition
A region \( R \) is **closed** if it contains all of its boundary points. A region \( R \) is **open** if it contains none of its boundary points.
Definition
A function $f(x, y)$ is continuous on a region $R$ if it is continuous at each point in $R$.

Remark: we need only evaluate limits along paths which lie in $R$. 

Example (1 of 2)

Determine where the following function is continuous.

\[ f(x, y) = \frac{3x^2y}{x^2 + y^2} \]
Example (1 of 2)

Determine where the following function is continuous.

\[ f(x, y) = \frac{3x^2y}{x^2 + y^2} \]

If \((a, b) \neq (0, 0)\) then

\[ \lim_{(x, y) \to (a, b)} \frac{3x^2y}{x^2 + y^2} = \frac{3a^2b}{a^2 + b^2} = f(a, b) \]

which implies that \(f(x, y)\) is continuous for all \((x, y) \neq (0, 0)\).
Example (1 of 2)

Determine where the following function is continuous.

\[ f(x, y) = \frac{3x^2y}{x^2 + y^2} \]

If \((a, b) \neq (0, 0)\) then

\[
\lim_{{(x,y) \to (a,b)}} \frac{3x^2y}{x^2 + y^2} = \frac{3a^2b}{a^2 + b^2} = f(a, b)
\]

which implies that \(f(x, y)\) is continuous for all \((x, y) \neq (0, 0)\). Since \(f(x, y)\) is undefined at \((0, 0)\), then \(f\) has a discontinuity at \((0, 0)\).
Determine where the following function is continuous.

\[ f(x, y) = \begin{cases} \frac{3x^2y}{x^2+y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases} \]
Example (2 of 2)

Determine where the following function is continuous.

\[ f(x, y) = \begin{cases} 
\frac{3x^2y}{x^2+y^2} & \text{if } (x, y) \neq (0, 0), \\
0 & \text{if } (x, y) = (0, 0).
\end{cases} \]

We saw in the last example that if \((x, y) \neq (0, 0)\) then \(f\) is continuous at \((x, y)\).
Example (2 of 2)

Determine where the following function is continuous.

\[ f(x, y) = \begin{cases} 
\frac{3x^2y}{x^2+y^2} & \text{if } (x, y) \neq (0, 0), \\
0 & \text{if } (x, y) = (0, 0).
\end{cases} \]

We saw in the last example that if \((x, y) \neq (0, 0)\) then \(f\) is continuous at \((x, y)\).
We have also seen in a previous example that

\[ \lim_{(x,y) \to (0,0)} \frac{3x^2y}{x^2 + y^2} = 0 \]
Example (2 of 2)

Determine where the following function is continuous.

\[ f(x, y) = \begin{cases} 
\frac{3x^2 y}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0), \\
0 & \text{if } (x, y) = (0, 0). 
\end{cases} \]

We saw in the last example that if \((x, y) \neq (0, 0)\) then \(f\) is continuous at \((x, y)\).

We have also seen in a previous example that

\[
\lim_{(x, y) \to (0, 0)} \frac{3x^2 y}{x^2 + y^2} = 0 = f(0, 0).
\]
Example (2 of 2)

Determine where the following function is continuous.

\[
f(x, y) = \begin{cases} 
\frac{3x^2y}{x^2+y^2} & \text{if } (x, y) \neq (0, 0), \\
0 & \text{if } (x, y) = (0, 0).
\end{cases}
\]

We saw in the last example that if \((x, y) \neq (0, 0)\) then \(f\) is continuous at \((x, y)\).

We have also seen in a previous example that

\[
\lim_{(x,y) \to (0,0)} \frac{3x^2y}{x^2 + y^2} = 0 = f(0, 0).
\]

So this piecewise defined function is also continuous at \((0, 0)\). We can say that \(f(x, y)\) is continuous for all \((x, y) \in \mathbb{R}^2\).
Composition of Functions

Theorem
Suppose that $f(x, y)$ is continuous at $(a, b)$ and $g(x)$ is continuous at the point $f(a, b)$. Then

$$h(x, y) = (g \circ f)(x, y) = g(f(x, y))$$

is continuous at $(a, b)$. 
Example

Where is the function $h(x, y) = \tan^{-1}(y/x)$ continuous?
Limits in Three Dimensions

Definition
Let \( f(x, y, z) \) be defined on the interior of a sphere centered at the point \((a, b, c)\), except possibly at \((a, b, c)\) itself. We say that

\[
\lim_{(x,y,z) \to (a,b,c)} f(x, y, z) = L
\]

if for every \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that

\[
|f(x, y, z) - L| < \epsilon
\]
whenever

\[
0 < \sqrt{(x - a)^2 + (y - b)^2 + (z - c)^2} < \delta.
\]
Definition

Let \( f(x, y, z) \) be defined on the interior of a sphere centered at the point \((a, b, c)\). We say that \( f \) is continuous at \((a, b, c)\) if

\[
\lim_{{(x, y, z) \to (a, b, c)}} f(x, y, z) = f(a, b, c).
\]

If \( f \) is not continuous at \((a, b, c)\), then we call \((a, b, c)\) a discontinuity of \( f \).
Example

Find the following limit, if it exists.

$$\lim_{{(x,y,z) \to (0,0,0)}} \frac{x^2 + 2y^2 + 3z^2}{x^2 + y^2 + z^2}$$
Solution

Consider the limits along the paths suggested below.

\[
\begin{align*}
\lim_{(t, 0, 0) \to (0, 0, 0)} t^2 + 0 + 0 &= 1 \\
\lim_{(0, t, 0) \to (0, 0, 0)} 0 + 2t^2 + 0 &= 2
\end{align*}
\]

Thus the limit does not exist.
Solution

Consider the limits along the paths suggested below.

\[ x(t) = t \quad y(t) = 0 \quad z(t) = 0 \]

versus

\[ x(t) = 0 \quad y(t) = t \quad z(t) = 0 \]

\[
\lim_{(t,0,0) \to (0,0,0)} \frac{t^2 + 0 + 0}{t^2 + 0 + 0} = 1
\]

\[
\lim_{(0,t,0) \to (0,0,0)} \frac{0 + 2t^2 + 0}{0 + t^2 + 0} = 2
\]

Thus the limit does not exist.
Example (2 of 2)

Determine the largest subset of three-dimensional space on which the function

\[ f(x, y, z) = \frac{x \cdot y \cdot z}{x^2 + y^2 - z} \]

is continuous.
Homework

- Read Section 12.2.
- Exercises: 1–43 odd